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CERTAIN SUBGROUPS OF HOMOTOPY GROUPS OF A TRANSFORMATION GROUP

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Introduction.

F. Rhodes [8] introduced the homotopy groups of a transformation group (X, G) as a generalization of the homotopy groups of a topological space X. D. H. Gottlieb [3] introduced the evaluation subgroups of the homotopy groups of a topological space.

In this paper, the concepts of certain subgroups of the homotopy groups of a space are generalized to those of the certain subgroups of the homotopy groups of a transformation group (X, G).

In section 1, we define and study the evaluation subgroup $E(X, x_0, G)$ of the fundamental group of a transformation group. The relation between evaluation maps from mapping space X^X to X and $E(X, x_0, G)$ is examined and it is shown that the evaluation subgroup is an invariant of the homotopy type of the transformation group. In addition a theorem on the evaluation subgroup of the product of two transformation groups is proved and the representation of $E(X, x_0, G)$ in terms of Gottlieb's group $G(X, x_0)$ is investigated.

In section 2, the evaluation subgroups $E_n(X, x_0, G)$ of the homotopy groups of a transformation group are defined and some results on $E(X, x_0, G)$ extend immediately to $E_n(X, x_0, G)$.

Next, we construct a subgroup $G_n(X, x_0, G)$ of $E_n(X, x_0, G)$ which is abelian (in fact, the subgroup of the center of the homotopy group of a transformation group). Finally, we show that if (X, G) is a H-transformation group, then $G_n(X, x_0, G)$ is equal to the homotopy group of a transformation group.

1. The evaluation subgroup $E(X, x_0, G)$.

Let (X, G) be a transformation group, where X is a topological space

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with x_0 as a base point. Given any element g of G, *a path f of order g* with base-point x_0 is a continuous map $f: I\to X$ such that $f(0)=x_0$ and $f(1)=gx_0$. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1+g_1f_2$ of order g_1g_2 defined by the equations

$$(f_1+g_1f_2)(s)=f_1(2s),\ \ 0\leq s\leq 1/2$$
$$(f_1+g_1f_2)(s)=g_1f_2(2s-1),\ \ 1/2\leq s\leq 1.$$

Two paths f and f' of the same order g are said to be *homotopic* if there is a continuous map $F: I^2\to X$ such that

$$F(s,0)=f(s),\ \ 0\leq s\leq 1,$$
$$F(s,1)=f'(s),\ \ 0\leq s\leq 1,$$
$$F(0,t)=x_0,\ \ 0\leq t\leq 1,$$
$$F(1,t)=gx_0,\ \ 0\leq t\leq 1.$$

The homotopy class of a path f of order g was denoted by $[f;g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0=g_2x_0$. F. Rhodes showed that the set of homotopy classes of paths of prescribed order with the rule of composition $*$ is a group, where $*$ is defined by $[f_1;g_1]*[f_2;g_2]=[f_1+g_1f_2;g_1g_2]$. This group was denoted by $\sigma(X,x_0,G)$, and was called the fundamental group of (X,G) with base point x_0.

In [2], a homotopy $H: X\times I\to X$ is called a *cyclic homotopy* if

$$H(x,0)=H(x,1)=x.$$

This concept of a topological space is generalized to that of a transformation group. A continuous map $H: X\times I\to X$ is called a *homotopy of order g* if

$$H(x,0)=x,\ \ H(x,1)=gx,$$

where g is an element of G. In another notation, h_t is a homotopy of order g if $h_0=1_X$ and $h_1=g$, where 1_X denotes the identity map of X. If h_t is a homotopy of order g, then the path $f: I\to X$ such that $f(t)=h_t(x_0)$ will be called *the trace of h_t*. The trace is a path of order g. In particular, if the acting group G is trivial, then the homotopy of order g is the cyclic homotopy which was defined by D. H. Gottlieb in [2].

DEFINITION. Let $E(X,x_0,G)$ be the set of all elements $[f:g]\in\sigma(X,x_0,G)$ such that f is the trace of a homotopy of order g, where $g\in G$.

For every $[f:g]\in E(X,x_0,G)$, there is at least one homotopy $F: X\times I\to X$ of order g with trace f', where $[f':g]=[f:g]$. We shall always assume that representations of homotopy classes in $E(X,x_0,G)$ are traces of homotopy of prescribed order.

THEOREM 1.1. *$E(X,x_0,G)$ is a subgroup of $\sigma(X,x_0,G)$.*

Proof. Let $[f_1:g_1]$ and $[f_2:g_2]$ be any two elements of $E(X,x_0,G)$.

Let h_t and k_t be the homotopies of order g_1, g_2 with trace f_1 and f_2 respectively. Define a homotopy $r_t : X \to X$ such that $r_t(x) = h_{2t}(x)$ for $0 \leq t \leq 1/2$ and $r_t(x) = g_1 k_{2t-1}(x)$ for $1/2 \leq t \leq 1$. Then we have $r_0(x) = x$ and $r_1(x) = g_1 g_2(x)$. The trace of r_t is the path $f_1 + g_1 f_2$ of order $g_1 g_2$. Hence $[f_1 : g_1] * [f_2 : g_2] = [f_1 + g_1 f_2 : g_1 g_2] \in E(X, x_0, G)$.

If $[f : g] \in E(X, x_0, G)$, then there exists a homotopy h_t of order g such that $h_0 = 1_X$ and $h_1 = g$ and $h_t(x_0) = f(t)$. If we take a homotopy $h'_t = g^{-1} h_{1-t} : X \to X$, then $h'_0 = 1_X$ and $h'_1 = g^{-1}$. Since $g^{-1} f \rho$ is the trace of h'_t, $[f : g]^{-1} = [g^{-1} f \rho : g]$ belongs to $E(X, x_0, G)$.

Note that $[f : g]$ may have two homotopies of order g which are not homotopic.

THEOREM 1. 2. *Let X be a pathwise connected CW-complex. If f is the trace of a homotopy H of order g and $[f' : g] = [f : g]$, then there exists a homotopy H' of order g with trace f' such that H' is homotopic to H.*

Proof. Let $J : I \times I \to X$ be a homotopy such that $J(s, 0) = f(s)$, $J(s, 1) = f'(s)$ and $J(0, t) = x_0$, $J(1, t) = g x_0$. Let L be the subcomplex of $X \times I$ given by $(X \times 0) \cup (X \times 1) \cup (x_0 \times I)$. Define a partial homotopy of H on L as follows; $k_t : L \to X$ such that $k_t(x, 0) = x$, $k_t(x, 1) = gx$, $k_t(x_0, s) = J(s, t)$. By the AHEP, there exists a homotopy $\boldsymbol{k}_t : X \times I \to X$ such that $\boldsymbol{k}_0 = H$, $\boldsymbol{k}_t / L = k_t$. Then $\boldsymbol{k}_1 : X \times I \to X$ is a homotopy of order g with trace f'.

In [8], F. Rhodes showed that if λ is a path from x_0 to x_1, then λ induces an isomorphism $\lambda_* : \sigma(X, x_0, G) \to \sigma(X, x_1, G)$ such that $\lambda_*[f : g] = [\lambda\rho + f + g\lambda : g]$.

THEOREM 1. 3. *Assume that X is a pathwise connected CW-complex. If λ is a path from x_0 to x_1, then λ induces an isomorphism $\lambda_* : E(X, x_0, G) \to E(X, x_1, G)$.*

Proof. Since λ_* is 1-1, all we must show is that $\lambda_*(E(X, x_0, G)) \subset E(X, x_1, G)$. Let $[f : g] \in E(X, x_0, G)$. Then there exists a homotopy $H : X \times I \to X$ of order g with trace f. By the homotopy extension property, there is a homotopy $J : X \times I \to X$ such that $J(x, 0) = x$ and $J(x_1, t) = \lambda\rho(t)$. Define $K : X \times I \to X$ by

$$K(x, t) = \begin{cases} J(x, 3t), & 0 \leq t \leq 1/3 \\ H(J(x, 1), 3t-1), & 1/3 \leq t \leq 2/3 \\ gJ(x, 3(1-t)), & 2/3 \leq t \leq 1. \end{cases}$$

Now K is a homotopy of order g such that $K(x_1, t) = (\lambda\rho + f + g\lambda)(t)$. So $\lambda_*[f : g] = [\lambda\rho + f + g\lambda : g] \in E(X, x_1, G)$.

In [2], D. H. Gottlieb defined a subgroup $G(X, x_0)$ of $\pi_1(X, x_0)$. If we take a map $i_*: G(X, x_0) \to E(X, x_0, G)$ such that $i_*[f]=[f: e]$, then we can identify $G(X, x_0)$ as a subgroup of $E(X, x_0, G)$.

THEOREM 1.4. *$G(X, x_0)$ is a normal subgroup of $E(X, x_0, G)$.*

Proof. Clearly $G(X, x_0)$ is a subgroup of $E(X, x_0, G)$. Let $[f: g]$ be any element of $E(X, x_0, G)$ and let $[\alpha : e]$ be any element of $G(X, x_0)$. Then there exist a homotopy $H: X\times I \to X$ of order g with trace f and a cyclic homotopy $K: X\times I \to X$ such that $K(x_0, t)=\alpha(t)$. Define $F: X\times I\to X$ by

$$F(x,t)=\begin{cases} H(x, 3t), & 0\leq t\leq 1/3 \\ gK(x, 3t-1), & 1/3\leq t\leq 2/3 \\ H(x, 3-3t), & 2/3\leq t\leq 1 \end{cases}$$

Then F is a cyclic homotopy such that $F(x_0, t)=(f+g\alpha+f\rho)(t)$. So $[f : g]*[\alpha : e]*[f: g]^{-1}=[f+g\alpha+f\rho : e]\in G(X, x_0)$. Thus $G(X, x_0)$ is a normal subgroup of $E(X, x_0, G)$.

In [8], a category mapping $(\varphi, \phi) : (X, G)\to(Y, H)$ consists of a continuous map $\varphi: X\to Y$ and a homomorphism $\phi: G\to H$ such that $\varphi(gx)=(\phi(g))(\varphi(x))$.

Let (X, G, π) be a transformation group. Let X^X be the space of continuous mappings from X to X with compact-open topology. Define $\pi' : X^X\times G\to X^X$ by $\pi'(f, g)=gf$, where $(gf)(x)=\pi(f(x), g)$ and $g\in G$, $f\in X^X$. Then π' is well-defined and continuous. Since $\pi'(f, e)=ef=f$ and $\pi'(\pi'(f, h), g)=\pi'(hf, g)=\pi'(f, gh)$, where e is the unit element of G and g, $h\in G$ and $f\in X^X$, (X^X, G, π') is a transformation group. Let $p: X^X\to X$ be the evaluation $p(f)=f(x_0)$. Assume that X is a locally compact. Then the evaluation map p is continuous. Since $p(gf)=gf(x_0)=g(f(x_0))=gp(f)$, where $g\in G$ and $f\in X^X$, $p=(p, 1_G) : (X^X, G)\to(X, G)$ is a category mapping. Thus $p_* : \sigma(X^X, 1_X, G)\to\sigma(X, x_0, G)$ is a homomorphism by $p_*[f; g]=[pf; g]$.

REMARK. There is a natural homeomorphism between the space of maps $(X^X)^I$ and $X^{X\times I}$ given by $\phi: (X^X)^I\to X^{X\times I}$ such that $\phi(f)(x, s)=(f(s))(x)$ for $x\in X$ and $s\in I$. Note that $f\sim f'$ iff $\phi(f)\sim\phi(f')$.

Because of the following theorem, $E(X, x_0, G)$ is called the evaluation subgroup of (X, G).

THEOREM 1.5. *Let X be a pathwise connected CW-complex. Then $p_*\sigma(X^X, 1_X, G)=E(X, x_0, G)$.*

Proof. By the remark, the path $f: I\to X^X$ of order g with base point 1_X corresponds to the homotopy $\phi(f) : X\times I\to X$ of order g. Since $p\circ f(t)=$

$(f(t))(x_0)=\phi(f)(x_0,t)$, $p\circ f=\phi(f)/I$. For every element $[f:g]\in\sigma(X^X, 1_X, G)$, $p_*[f:g]=[p\circ f:g]$ and there exists a homotopy $\phi(f)$ of order g whose trace equals $p\circ f$. Thus $p_*[f:g]\subset E(X,x_0,G)$. Conversely, for each element $[f:g]$ of $E(X,x_0,G)$, there exists a homotopy $F:X\times I\to X$ of order g with trace f. Then $\phi^{-1}(F)$ is a path of order g with base point 1_X in X^X. So $[\phi^{-1}(F):g]$ belongs to $\sigma(X^X,1_X,G)$ but $p\circ\phi^{-1}(F)(s)=(\phi^{-1}(F)(s))(x_0)=F(x_0,s)=f(s)$. Therefore $[f:g]\in p_*\sigma(X^X,1_X,G)$.

A category mapping $(\varphi,\psi):(X,G)\to(Y,H)$ has a *right homotopy inverse* if there is a category mapping $(\varphi',\psi'):(Y,H)\to(X,G)$ such that $\varphi\varphi'$ is homotopic to 1_Y.

THEOREM 1.6. *Suppose that X is a pathwise connected space and Y is a pathwise connected CW-complex. If $(\varphi,\psi):(X,G)\to(Y,H)$ has a right homotopy inverse $(\varphi',\psi'):(Y,H)\to(X,G)$, then $(\varphi,\psi)_*(E(X,x_0,G))\subset E(Y,\varphi x_0,H)$.*

Proof. Choose a base point x_0 in X, and let $\varphi x_0=y_0$, $\varphi' y_0=x_1$ and $\varphi x_1=y_1$. Let $[f:g]\in E(X,x_0,G)$ and let $H:X\times I\to X$ be a homotopy of order g with trace f. Define a map $K:Y\times I\to Y$ by $K(y,t)=\varphi H(\varphi'(y),t)$. Then $K(y,0)=\varphi\varphi'(y)$, $K(y,1)=(\psi g)(\varphi\varphi'(y))$ and $K(y_0,t)=\varphi H(x_1,t)$. Since $\varphi\varphi'$ is homotopic to 1_Y, there is a homotopy $J:Y\times I\to Y$ such that $J(y,0)=\varphi\varphi'(y)$ and $J(y,1)=y$. Define a homotopy $T:Y\times I\to Y$ by

$$T(y,t)=\begin{cases}J(y,1-3t), & 0\leq t\leq 1/3\\ K(y,3t-1), & 1/3\leq t\leq 2/3\\ \psi gJ(y,3t-2), & 2/3\leq t\leq 1.\end{cases}$$

Then T is a homotopy of order ψg on Y. Let $\alpha:I\to Y$ be the path given by $\alpha(t)=J(y_0,t)$ and let τ be the trace of T at y_0. Then $\tau(t)=T(y_0,t)=(\alpha\rho+\varphi H(x_1,\ )+\psi g\alpha)(t)$. Let λ be a path from x_0 to x_1. Define a map $P:I\times I\to Y$ by $P(s,t)=\varphi H(\lambda\rho(t),s)$. Then $P(s,0)=\varphi H(x_1,s)$, $P(s,1)=\varphi f(s)$ and $P(0,t)=\varphi\lambda\rho(t)$, $P(1,t)=\psi g\varphi\lambda\rho(t)$. Define $Q:I\times I\to Y$ by

$$Q(s,t)=\begin{cases}\varphi\lambda\rho(2s), & 0\leq s\leq\dfrac{1-t}{2}\\ P\left(\dfrac{4s+2t-2}{3t+1},\ 1-t\right), & \dfrac{1-t}{2}\leq s\leq\dfrac{t+3}{4}\\ \psi g\varphi\lambda(4s-3), & \dfrac{t+3}{4}\leq s\leq 1\end{cases}$$

Then Q is a homotopy with fixed end-points from $\varphi\lambda\rho+\varphi f+\psi g\varphi\lambda$ to $\varphi H(x_1,\ )$. Thus $[\tau:\psi g]=[\alpha\rho+\varphi H(x_1,\ )+\psi g\alpha:\psi g]=[\alpha\rho+\varphi\lambda\rho+\varphi f+\psi g\varphi\lambda+\psi g\alpha:\psi g]=[(\varphi\lambda+\alpha)\rho:e]*[\varphi f:\psi g]*[\varphi\lambda+\alpha:e]$. Since $[\tau:\psi g]\in E(Y,y_0,H)$ and $\varphi\lambda+\alpha$ is a loop at y_0, we obtain $[\varphi f:\psi g]=((\varphi\lambda+\alpha)\rho)_*[\tau:\psi g]\in E(Y,y_0,H)$. Thus $(\varphi,\psi)_*(E(X,x_0,G))\subset E(Y,y_0,H)$.

Let $(\varphi, \phi) : (X, G) \to (Y, H)$ be a category mappping. We say that (φ, ϕ) has a *left homotopy inverse* if there is a category map $(\varphi', \phi') : (Y, H) \to (X, G)$ such that $\varphi'\varphi$ is homotopic to 1_X.

THEOREM 1.7. *If* $(\varphi, \phi) : (X, G) \to (Y, H)$ *has a left homotopy inverse* $(\varphi'\phi') : (Y, H) \to (X, G)$, *then* $(\varphi, \phi)_*[f : g] \in E(Y, y_0, H)$ *implies that* $[f : g] \in E(X, x_0, G)$ *where* $\varphi(x_0) = y_0$.

Proof. Since $(\varphi, \phi)_*[f : g] = [\varphi f : \phi g] \in E(Y, y_0, H)$, there exists a homotopy of order ϕg, $K : Y \times I \to Y$, such that $K(y, 0) = y$, $K(y, 1) = (\phi g)y$ and $K(y_0, t) = \varphi f(t)$. Define a homotopy $H : X \times I \to X$ by $H(x, t) = \varphi' K(\varphi(x), t)$. Then $H(x, 0) = \varphi'\varphi(x)$, $H(x, 1) = (\phi'\phi g)(\varphi'\varphi)(x)$ and $H(x_0, t) = \varphi'\varphi f(t)$. Since $\varphi'\varphi \sim 1_X$, there is a homotopy $J : X \times I \to X$ such that $J(x, 0) = \varphi'\varphi(x)$ and $J(x, 1) = x$. Define a homotopy $T : X \times I \to X$ by

$$T(x, t) = \begin{cases} J(x, 1-3t), & 0 \le t \le 1/3 \\ H(x, 3t-1), & 1/3 \le t \le 2/3 \\ J(gx, 3t-2), & 2/3 \le t \le 1. \end{cases}$$

Then T is a homotopy of order g on X. Let $\alpha : I \to X$ be the path given by $\alpha(t) = J(x_0, t)$ and let τ be the trace of T at x_0. Then $\tau(t) = T(x_0, t) = (\alpha\rho + \varphi'\varphi f + J(gx_0,))(t)$. Define a homotopy $P : I \times I \to X$ by $P(s, t) = J(f(s), t)$. Then $P(s, 0) = \varphi'\varphi f(s)$, $P(s, 1) = f(s)$ and $P(0, t) = \alpha(t)$ and $P(1, t) = J(gx_0, t)$. Define $Q : I \times I \to X$ by

$$Q(s, t) = \begin{cases} \alpha\rho(2s), & 0 \le s \le \frac{1-t}{2} \\ P\left(\frac{4s+2t-2}{3t+1}, t\right), & \frac{1-t}{2} \le s \le \frac{t+3}{4} \\ J(gx_0, 4s-3), & \frac{t+3}{4} \le s \le 1 \end{cases}$$

Then Q is a homotopy with fixed end points from $\alpha\rho + \varphi'\varphi f + J(gx_0,)$ to f. Thus $[\tau : g] = [\alpha\rho + \varphi'\varphi f + J(gx_0,) : g] = [f : g] \in E(X, x_0, G)$.

In [8], two transformation groups (X, G) and (Y, H) are said to be of the *same homotopy type* if there exist category maps $(\varphi, \phi) : (X, G) \to (Y, H)$ and $(\varphi'\phi') : (Y, H) \to (X, G)$ such that ϕ and ϕ' are isomorphisms and $\varphi'\varphi$ and $\varphi\varphi'$ are homotopic to the identity maps of X and Y, respectively.

THEOREM 1.8. *Suppose that* X *is a pathwise connected and* Y *is a pathwise connected* CW*-complex. If* $(\varphi, \phi) : (X, G) \to (Y, H)$ *is a homotopy equivalence, then* $(\varphi, \phi)_*$ *carries* $E(X, x_0, G)$ *isomorphically onto* $E(Y, \varphi x_0, H)$.

Proof. Choose a base point x_0 in X, and let $\varphi x_0 = y_0$. In [8], it was proved that if $(\varphi, \phi) : (X, G) \to (Y, H)$ is a homotopy equivalence, then

$(\varphi, \psi)_* : \sigma(X, x_0, G) \to \sigma(Y, y_0, H)$ is an isomorphism. Thus we need to show that $(\varphi, \psi)_*(E(X, x_0, G)) = E(Y, y_0, H)$.

By Theorem 1.6, $(\varphi, \psi)_* \; E(X, x_0, G) \subset E(Y, y_0, H)$ and by Theorem 1.7, $(\varphi, \psi)_*^{-1} E(Y, y_0, H) \subset E(X, x_0, G)$. So $(\varphi, \psi)_* \; E(X, x_0, G) = E(Y, y_0, H)$.

THEOREM 1.9. *Let (X, G) and (Y, H) be transformation groups, and let X and Y are pathwise connected CW-complexes. Then* $E(X \times Y, (x_0, y_0), G \times H) \cong E(X, x_0, G) \oplus E(Y, y_0, H)$.

Proof. Let $Z = X \times Y$ and $z_0 = (x_0, y_0)$. In [8], there eixsts an isomorphism $\theta_* : \sigma(X, x_0, G) \oplus \sigma(Y, y_0, H) \to (Z, z_0, G \times H)$ such that $\theta_*([f_x : g], [f_y : h]) = [\theta(f_x, f_y) : (g, h)]$ where

$$\theta(f_x, f_y) = \begin{cases} (f_x(2t), y_0), & 0 \le t \le 1/2 \\ (gx_0, f_y(2t-1)), & 1/2 \le t \le 1 \end{cases}$$

Note that $\theta_*^{-1} : \sigma(Z, z_0, G \times H) \to \sigma(X, x_0, G) \oplus \sigma(Y, y_0, H)$ is an isomorphism such that $\theta_*^{-1}([f : (g, h)]) = (\varphi_x, P_1)_*[f : (g, h)] \oplus (\varphi_y, P_2)_*[f : (g, h)]$, where P_1 and P_2 are the projections of $G \times H$ onto G and H respectively and φ_x and φ_y are the projections of Z onto X and Y respectively.

Now $\theta_*^{-1} E(Z, z_0, G \times H) \subset E(X, x_0, G) \oplus E(Y, y_0, H)$ as may readily be seen by noting that projections (φ_x, P_1) and (φ_y, P_2) have right homotopy inverse respectively and applying Theorem 1.6. On the other hand, let $[f_x : g]$ and $[f_y : h]$ be elements of $E(X, x_0, G)$ and $E(Y, y_0, H)$ respectively. Now $\theta_*([f_x : g], [f_y : h]) = [\theta(f_x, f_y) : (g, h)]$. Since $[f_x : g] \in E(X, x_0, G)$ and $[f_y : h] \in E(Y, y_0, H)$, there exist homotopies $H : X \times I \to X$ and $K : Y \times I \to Y$ of order g and h with traces f_x and f_y respectively. Let $J : X \times Y \times I \to X \times Y$ be defined as follows;

$$J(x, y, t) = \begin{cases} H(x, 2t), & 0 \le t \le 1/2 \\ (gx, K(y, 2t-1)), & 1/2 \le t \le 1. \end{cases}$$

It can easily be verified that J is a homotopy of order (g, h) on $X \times Y$ with trace $\theta(f_x, f_y)$. Hence $\theta_*([f_x : g], [f_y : h]) = [\theta(f_x, f_y) : (g, h)] \in E(Z, z_0, G \times H)$. So $\theta_*(E(X, x_0, G) \oplus E(Y, y_0, H)) \subset E(Z, z_0, G \times H)$.

Let (X, G) be a transformation group and M be the path component of the space of maps from X to X containing the identity map. Let $G_0 = M \cap G$. Then G_0 is a subgroup of G, and $E(X, x_0, G)$ is a subgroup of $\sigma(X, x_0, G_0)$. A family $K = \{k_g : g \in G_0\}$ of paths in X is said to be *a family of preferred traces at* x_0 if for each element g of G_0, k_g is the trace of a homotopy of order g, if $k_e \sim x_0'$, and if for every pair of elements g_1, g_2 of G_0, $k_{g_1 g_2} \sim k_{g_1} + g_1 k_{g_2}$. However, if the group G_0 is free then for every generator g, the trace k_g of a homotopy of order g can be chosen by $k_g(t) = \phi(k)(x_0, t)$, where k is a path from 1_X to g in X^X and ϕ is a natural homeomorphism from $(X^X)^I$ to $X^{X \times I}$. The equation $k_{g_1 g_2} = k_{g_1} + g_1 k_{g_2}$ can then be used to define a family of preferred traces at x_0.

THEOREM 1.10. *Let (X, G) admits a family of preferred traces at x_0. Then $E(X, x_0, G)=G(X, x_0)\oplus G_0$, where $G(X, x_0)$ is the Gottlieb group.*

Proof. Define a map $j: E(X, x_0, G)\to G$ by $j([f:g])=g$. For every $g\in G_0$, there exists a path k from 1_X to g in X^X and $\phi(k)$ is a homotopy of order g with trace $p\circ k$. So $[p\circ k: g]\in E(X, x_0, G)$. On the other hand, let $[f:g]\in E(X, x_0, G)$. Then there exists a homotopy H of order g with trace f and $\phi^{-1}(H)$ is a path from 1_X to g in X^X. So $g\in G_0$. Thus j is an epimorphism. Then there exists a short exact sequence $0\to G(X, x_0)\xrightarrow{i} E(X, x_0, G)\xrightarrow{j} G_0\to 0$. Let $p: G_0\to E(X, x_0, G)$ be given by $p(g)=[k_g: g]$. Then p is a homomorphism and $j\circ p=1_{G_0}$.

COROLLARY 1.11. *If (X, G) admits a family of preferred traces at x_0 and G is abelian, then $E(X, x_0, G)$ is an abelian subgroup of $\sigma(X, x_0, G)$.*

2. Abelian subgroups in homotopy group $\sigma_n(X, x_0, G)$.

In [9], the homotopy groups $\sigma_n(X, x_0, G)$ of a transformation group (X, G) are defined. By identifying corresponding points on all but the first pair of opposite faces of the cube I^n we obtain as a quotient space the n-dimensional cylinder C^n. It will be convenient to use the notation $\hat{t}_i=t_i(\text{mod } 1)$, $C^n=\{(t_1, \hat{t}_2, \ldots, \hat{t}_n)\}\, 0\le t_i\le 1\}$. If $r<n$, a map $f: C^r\to X$ gives rise to a map $f^n: C^n\to X$ defined by $f^n(t_1, \hat{t}_2, \ldots, \hat{t}_n)=f(t_1, \hat{t}_2, \ldots, \hat{t}_r)$. A continuous map $f: C^n\to X$ is called *a map of C^n of order g* if $f(0, \hat{t}_2\ldots, \hat{t}_n)=x_0$ and $f(1, \hat{t}_2, \ldots, \hat{t}_n)=gx_0$. The homotopy group $\sigma_n(X, x_0, G)$ is the group of homotopy classes of maps of C^n of prescribed order with a rule of composition. For a map f of C^n of order g, a continuous map $H: X\times C^n\to X$ will be called *an associated map for f* if $H(x, 0, \hat{t}_2, \ldots, \hat{t}_n)=x$, $H(x, 1, \hat{t}_2, \ldots, \hat{t}_n)=gx$ and $H(x_0, t_1, \hat{t}_2, \ldots, \hat{t}_n)=f(t_1, \hat{t}_2, \ldots, \hat{t}_n)$. We can define the n-th evaluation subgroup $E_n(X, x_0, G)$ of $\sigma_n(X, x_0, G)$ as follows; $E_n(X, x_0, G)=\{[f:g]\in\sigma_n(X, x_0, G)\,|\, f$ has an associated map$\}$. Some results on $E(X, x_0, G)$ extend immediately to $E_n(X, x_0, G)$. That is, $E_n(X, x_0, G)$ is a subgroup of $\sigma_n(X, x_0, G)$ and if λ is a path from x_0 to x_1, then λ induces an isomorphism $\lambda_*: E_n(X, x_0, G)\to E_n(X, x_1, G)$ by $\lambda_*[f:g]=[\lambda^n\rho^n+f+g\lambda^n: g]$, where X is a pathwise connected CW-complex. Clearly $p_*(\sigma_n(X^X, 1_X, G))=E_n(X, x_0, G)$ where $p: X^X\to X$ is the evaluation $p(f)=f(x_0)$. Also, the n-th evaluation subgroup is an invariant of homotopy type in the category of transformation groups.

We have not been able to answer whether $E_n(X, x_0, G)$ is an abelian subgroup of $\sigma_n(X. x_0, G)$ or not. However, we can construct a subgroup of $E_n(X, x_0, G)$ which is an abelian (in fact, the subgroup of the center of $\sigma_n(X, x_0, G)$).

From now G will be an abelian group. Let (X, G, π) be a transformation group and $\pi' : X\times C^n\times G\rightarrow X\times C^n$ be defined by $\pi'((x, t_1, t_2, ..., t_n), g)=((\pi(x, g), (t_1, \hat{t}_2, ..., \hat{t}_n))$. Then $(X\times C^n, G, \pi')$ is a transformation group. A continuous map $H : X\times C^n\rightarrow X$ is called a *homomorphism at* x_0 *of order* g if $H(x, 0, \hat{t}_2, ..., \hat{t}_n)=x$, $H(x, 1, \hat{t}_2, ..., \hat{t}_n)=gx$ and $H(g'x_0, t_1, \hat{t}_2, ..., \hat{t}_n)=g'H(x_0, t_1, \hat{t}_2, ..., \hat{t}_n)$ for all $g'\in G$. In this case, we define a map $f : C^n\rightarrow X$ such that $f(t_1, \hat{t}_2, ..., \hat{t}_n)=H(x_0, t_1, \hat{t}_2, ..., \hat{t}_n)$. Then f is called the *trace of H.*

DEFINITION. Let $G_n(X, x_0, G)$ be the set of all elements $[f : g]\in\sigma_n(X, x_0, G)$ such that there exists a homomorphism at x_0 of order g with trace f.

THEOREM 2. 1. *$G_n(X, x_0, G)$ is a subgroup of $\sigma_n(X, x_0, G)$.*

Proof. Let $[f_1 : g_1]$ and $[f_2 : g_2]$ be any two elements of $G_n(X, x_0, G)$. Let F and H be the homomorphisms at x_0 of order g_1, g_2 with trace f_1 and f_2 respectively. Define a map $K : X\times C^n\rightarrow X$ by

$$K(x, t_1, \hat{t}_2, ..., \hat{t}_n)=\begin{cases}F(x, 2t_1, \hat{t}_2, ..., \hat{t}_n), & 0\leq t_1\leq 1/2\\ g_1H(x, 2t_1-1, \hat{t}_2, ..., \hat{t}_n), & 1/2\leq t_1\leq 1.\end{cases}$$

Then K is a homomorphism at x_0 of order g_1g_2 with trace $f_1+g_1f_2$. Thus $[f_1 : g_1]*[f_2 : g_2]=[f_1+g_1f_2 : g_1g_2]\in G_n(X, x_0, G)$. Also $[f_1 : g_1]^{-1}\in G_n(X, x_0, G)$ since $[f_1 : g_1]^{-1}=[g_1^{-1}f_1\rho^n : g_1^{-1}]$ and $g_1^{-1}f_1\rho^n$ is the trace of $g_1^{-1}F(x, 1-t_1, \hat{t}_2, ..., \hat{t}_n)$.

Note that $G_n(X, x_0, G)$ is a subgroup of $E_n(X, x_0, G)$.

For $[f : g]\in\sigma_n(X, x_0, G)$ and $[f' : g']\in\sigma_m(X, x_0, G)$, a continuous map $F : C^m\times C^n\rightarrow X$ is called a *W-map of f for f'* if

$$\begin{aligned}
&F((0, \hat{s}_2, ..., \hat{s}_m),\ (t_1, \hat{t}_2, ..., \hat{t}_n))=f(t_1, \hat{t}_2, ..., \hat{t}_n)\\
&F((1, \hat{s}_2, ..., \hat{s}_m),\ (t_1, \hat{t}_2, ..., \hat{t}_n))=g'f(t_1, \hat{t}_2, ..., \hat{t}_n)\\
&F((s_1, \hat{s}_2, ..., \hat{s}_m),\ (0, \hat{t}_2, ..., \hat{t}_n))=f'(s_1, \hat{s}_2, ..., \hat{s}_m)\\
&F((s_1, \hat{s}_2, ..., \hat{s}_m),\ (1, \hat{t}_2, ..., \hat{t}_n))=gf'(s_1, \hat{s}_2, ..., \hat{s}_m),\\
&\text{where } (t_1, \hat{t}_2, ..., \hat{t}_n)\in C^n \text{ and } (s_1, \hat{s}_2, ..., \hat{s}_m)\in C^m.
\end{aligned}$$

DEFINITION. Let $W_n(X, x_0, G)$ be the set of all elements $[f : g]\in\sigma_n(X, x_0, G)$ such that for every m and for every $[f' : g']\in\sigma_m(X, x_0, G)$, there exists a W-map of f for f'.

THEOREM 2. 2. *$W_n(X, x_0, G)$ is a subgroup of $\sigma_n(X, x_0, G)$.*

Proof. Let $[f_1 : g_1]$ and $[f_2 : g_2]$ be any two elements of $W_n(X, x_0, G)$

and for every m, let $[f:g]\in\sigma_m(X,x_0,G)$. Let F and H be W-maps of f_1, f_2 for f respectively.

Define $K: C^m\times C^n\to X$ by

$$K((s_1,\hat{s}_2,\dots,\hat{s}_m),(t_1,\hat{t}_2,\dots,\hat{t}_n))=\begin{cases}F((s_1,\hat{s}_2,\dots,\hat{s}_m),\\ \quad(2t_1,\hat{t}_2,\dots,\hat{t}_n)),\ 0\le t_1\le 1/2\\ g_1H((s_1,\hat{s}_2,\dots,\hat{s}_m),(2t_1-1,\\ \quad\hat{t}_2,\dots,\hat{t}_n)),1/2\le t_1\le 1\end{cases}$$

Then K is a W-map of $f_1+g_1f_2$ for f. Thus $[f_1:g_1]*[f_2:g_2]=[f_1+g_1f_2:g_1g_2]\in W_n(X,x_0,G)$. Also $[f_1:g_1]^{-1}\in W_n(X,x_0,G)$ since $[f_1:g_1]^{-1}=[g_1^{-1}f_1\rho^n:g_1^{-1}]$and $g_1^{-1}F((s_1,\hat{s}_2,\dots,\hat{s}_m),(1-t_1,\hat{t}_2,\dots,\hat{t}_n))$ is a W-map of $g_1^{-1}f_1\rho^n$ for f.

THEOREM 2.3. *$G_n(X,x_0,G)$ is a subgroup of $W_n(X,x_0,G)$.*

Proof. Let $[f_1:g_1]\in G_n(X,x_0,G)$ and for every m, let $[f:g]\in\sigma_m(X,x_0,G)$. Let $F:X\times C^n\to X$ be a homomorphism at x_0 with trace f_1. Define $K:C^m\times C^n\to X$ by $K((s_1,\hat{s}_2,\dots,\hat{s}_m),(t_1,\hat{t}_2,\dots,\hat{t}_n))=F(f(s_1,\hat{s}_2,\dots,\hat{s}_m),(t_1,\hat{t}_2,\dots,\hat{t}_n))$. Then K is a W-map of f_1 for f. So $[f_1:g_1]\in W_n(X,x_0,G)$.

THEOREM 2.4. *$W_n(X,x_0,G)\subset Z(\sigma_n(X,x_0,G))$, the center of $\sigma_n(X,x_0,G)$.*

Proof. Let $[f:g]\in W_n(X,x_0,G)$ and $[f':g']\in\sigma_n(X,x_0,G)$. Let $F:C^n\times C^n\to X$ be a W-map of f for f'. Define a homotopy $K:C^n\times I\to X$ by

$$K((s_1,\hat{s}_2,\dots,\hat{s}_n),t)=\begin{cases}f'(2s_1,\hat{s}_2,\dots,\hat{s}_n),\quad 0\le s\le\frac{1-t}{2}\\ \left(F((1-t,\hat{s}_2,\dots,\hat{s}_n),\left(\frac{4s_1+2t-2}{3t+1},\hat{s}_2,\dots,\hat{s}_n\right)\right),\\ \quad\frac{1-t}{2}\le s_1\le\frac{t+3}{4}\\ gf'\rho^n(4s_1-3,\hat{s}_2,\dots,\hat{s}_n),\quad\frac{t+3}{4}\le s_1\le 1\end{cases}$$

The K is a homotopy with fixed end-points from $f'+g'f+gf'\rho^n$ to$f$. Thus $[f':g']*[f:g]*[f':g']^{-1}=[f'+g'f+gf'\rho^n:g]=[f:g]$. So $[f:g]\in Z(\sigma_n(X,x_0,G))$.

Let X be a CW-complex. It is easy to show that X is an H-space iff there exists a continuous map $\mu:X\times X\to X$ such that $\mu(x_0,x)=\mu(x,x_0)=x$ for all $x\in X$. A transformation group (X,G) is called a *H-transformation group* if there exists a continuous map $\mu:X\times X\to X$ such that $\mu(gx_0,x)=\mu(x,gx_0)=gx$ for every $g\in G$.

In fact, H-spaces are H-transformation groups with the trivial acting group.

THEOREM 2.5. *If (X,G) is an H-transformation group, then we have $G_n(X,x_0,G)=\sigma_n(X,x_0,G)$ for each $n\in N$.*

Proof. Let (X, G) be an H-transformation group and $\mu : X\times X\to X$ is a continuous map such that $\mu(gx_0, x)=\mu(x, gx_0)=gx$ for every $g\in G$.
For each n, let $[f : g]$ be any element of $\sigma_n(X, x_0, G)$.
Define $H : X\times C^n\to X$ by

$$H(x, t_1, \hat{t}_2, \dots, \hat{t}_n)=\mu(x, f(t_1, \hat{t}_2, \dots, \hat{t}_n))$$

Then H is a homomorphism at x_0 of order g with trace f. So $[f : g]\in G_n(X, x_0, G)$.

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