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CONDITIONAL YEH-WIENER INTEGRALS

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1. Introduction

Let $Q=[0,p]\times[0,q]$, where p and q are some fixed positive real numbers. Let $C_2[Q]$, called *Yeh-Wiener space*, denote the function space $\{x(\cdot,\cdot)\,|\,x(s,0)=x(0,t)=0,\ x(s,t)$ is a real valued continuous function on $Q\}$ with the uniform norm $\|x\|=\max\limits_{(s,t)\in Q}|x(s,t)|$. Let $\mathcal{I}$ be the algebra of all subsets of $C_2[Q]$ of the form

$$I=\{x\in C_2[Q]\,|\,(x(s_1,t_1),\cdots,x(s_m,t_n))\in E\}$$

where m and n are any positive integers, $0=s_0<s_1<\cdots<s_m=p$, $0=t_0<t_1<\cdots<t_n=q$ and E is an arbitrary Lebesgue measurable set in the mn-dimensional Euclidean space R^{mn}. Let $(C_2[Q],\ \mathcal{Y},\ m_y)$, called the *Yeh-Wiener measure space*, denote the complete probability space where $\mathcal{Y}$ is the σ-algebra of Caratheodory measurable subsets of $C_2[Q]$ with respect to the outer measure induced by the probability measure m_y on the algebra $\mathcal{I}$ defined for $I\in\mathcal{I}$ by

$$m_y(I)=\prod_{j=1}^{m}\prod_{k=1}^{n}\{2\pi(s_j-s_{j-1})(t_k-t_{k-1})\}^{-\frac{1}{2}}\cdot$$
$$\int_E \exp\left\{-\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{(u_{j,k}-u_{j-1,k}-u_{j,k-1}+u_{j-1,k-2})^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\right\}$$
$$du_{11}\cdots du_{mn}$$

where $u_{0,k}=u_{j,0}=u_{0,0}=0\,(j=1,2,\cdots,m,\ k=1,2,\cdots,n)$. (see [8]).

For a real valued Yeh-Wiener measurable (i.e. $\mathcal{Y}$-measurable) functional F on $C_2[Q]$, the Yeh-Wiener integral of F (i.e. the integral of F with respect to m_y) will be denoted by

$$E^y[F]=\int_{C_2[Q]}F(x)\,dm_y(x)$$

whenever the integral exists. We say that F is *Yeh-Wiener integrable* if $E^y[|F|]<\infty$. The Yeh-Wiener measurability and Yeh-Wiener integrability of a complex valued functional on $C_2[Q]$ are defined in terms of its real

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and imaginary parts.

In [10], Yeh introduced the notion of conditional Wiener integral which is meant the conditional expectation $E^w(Z|X)$ of a real or complex valued Wiener integrable functional Z conditioned by a Wiener measurable functional X on the Wiener space which is given as a function on the value space of X. The organization of this paper is as follows:

In Section 2 we investigate some properties of the Paley-Wiener-Zygmund ($P. W. Z.$) integral over Q by means of the stochastic integral and then prove a version of the $P. W. Z.$ Theorem for $C_2[Q]$ which is an extension of Theorem 29. 7 of [7] and the Yeh's result proved in [12; Theorem II].

In Section 3 we define, following Yeh [10], a conditional Yeh-Wiener integral and state three inversion formulae for conditional Yeh-Wiener integrals (Theorems 3. 3—3. 5) which are analogous to the corresponding formulae for conditional Wiener integrals. We then give several examples of evaluation of conditional Yeh-Wiener integrals.

2. Notes on the *P. W. Z.* integral over *Q*

Let Y be a real valued function on $Q\times C_2[Q]$ defined by

$$Y((s,t),x)=x(s,t) \ \ for \ \ ((s,t),x)\in Q\times C_2[Q].$$

Then Y is a measurable stochastic process on the probability space $(C_2[Q], \mathcal{Y}, m_y)$ and the domain Q in which the space of sample functions $Y(\cdot, x)$, $x\in C_2[Q]$, coincides with the sample space $C_2[Q]$. This stochastic process will be referred to as the *Yeh-Wiener process* on the domain Q. It is shown from the definition of $(C_2[Q], \mathcal{Y}, m_y)$ that $Y((s,t),\cdot)\sim N(0,st)$ (i. e. normally distributed with mean 0 and variance st) and $E^y(Y((s,t),\cdot)\cdot Y((u,v),\cdot))=\min\{s,u\}\cdot\min\{t,v\}$ for every $(s,t),(u,v)\in Q$.

DEFINITION 2. 1. A real valued function $f(s,t)$ is said to be *of bounded variation* on Q (for symbols, $f\in BV[Q]$) if the following are satisfied:

(i) $f(s,0)$ and $f(0,t)$ are of bounded variation on $[0,p]$ and $[0,q]$, respectively

(ii) The total variation $V(f)$ of f on Q is finite, where $V(f)$ is the supremum of

$$\sum_{j=1}^{n}\sum_{i=1}^{m}|f(s_i,t_j)-f(s_{i-1},t_j)-f(s_i,t_{j-1})+f(s_{i-1},t_{j-1})|$$

for any partition P: $0=s_0<s_1<\cdots<s_m=p$, $0=t_0<t_1<\cdots<t_n=q$.

It is known [4] that if $f\in BV[Q]$, then f is continuous almost everywhere on Q, so that f is Riemann integrable over Q.

For each $n=1,2,\cdots$, let P_n denote a partition of Q given by

$$0=s_{n,0}<s_{n,1}<\cdots<s_{n,p(n)}=p,\quad 0=t_{n,0}<t_{n,1}<\ldots<t_{n,q(n)}=q$$

with $\lim\limits_{n\to\infty}\|P_n\|=0$, where $\|P_n\|=\max\limits_{i,j}\{(s_{n,i}-s_{n,i-1}),\ (t_{n,j}-t_{n,j-1})\}$.

For each partition P_n of Q, let ξ_n denote a set of points in Q given by

$$\xi_n=\{(u_{n,i},v_{n,j})\in[s_{n,i-1},\ s_{n,i}]\times[t_{n,j-1},t_{n,j}]\mid i=1,2,\ldots,p(n),\ j=1,2,\ldots,q(n)\}.$$

For $f\in BV[Q]$, let $S(f,p_n,\xi_n)(x)$ denote the Riemann-Stieltjes sum of f with respect to Y defined by

$$S(f,P_n,\xi_n)(x)=\sum_{j=1}^{q(n)}\sum_{i=1}^{p(n)}f(u_{n,i},v_{n,j})\,\Delta^n{}_{i,j}Y(x),\quad x\in C_2[Q]$$

where $\Delta^n{}_{i,j}Y(x)=Y((s_{n,i},t_{n,j}),x)-Y((s_{n,i-1},t_{n,j}),x)$
$\qquad -Y((s_{n,i},t_{n,j-1}),x)+Y((s_{n,i-1},t_{n,j-1}),x)$

REMARK 2.1. We note that $S(f,P_n,\xi_n)(x)$ is, indeed, equal to the Riemann-Stieltjes sum of $f\in BV[Q]$ with respect to $x\in C_2[Q]$ corresponding to P_n and ξ_n, and that it is known [11] that the Riemann-Stieltjes integral $\int_Q f dx$ of $f\in BV[Q]$ exists for every $x\in C_2[Q]$. Hence $\lim\limits_{n\to\infty}S(f,\ P_n,\ \xi_n)(x)$ exists and is equal to $\int_Q f dx$ for every $x\in C_2[Q]$.

DEFINITION 2.2. For $f\in BV[Q]$, we define the *stochastic integral* $I(f)$ *of* f *with respect to the Yeh-Wiener process* Y is defined by

$$I(f)(x)=\lim_{n\to\infty}S(f,P_n,\xi_n)(x),\quad x\in C_2[Q].$$

We state the following lemma which can be proved by simple computations.

LEMMA 2.1. *With the same notation as in the above, we have the following*:

(1) $\Delta^n{}_{i,j}Y(\cdot)\sim N(0,(s_{n,i}-s_{n,i-1})(t_{n,j}-t_{n,j-1}))$ *for all* i,j *and* n.
(2) $E^y(\Delta^n{}_{i,j}Y(\cdot)\cdot\Delta^n{}_{k,l}Y(\cdot))=0$ *for all* $(i,j)\neq(k,l)$.

Let $L^2[C_2[Q]]$ ($L^2[Q]$) denote the real Hilbert space of all square Yeh-Wiener integrable functions over $C_2[Q]$ (Lebesgue integrable functions over Q, resp.). We shall write $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for inner product and norm in both spaces $L^2[C_2[Q]]$ and $L^2[Q]$ since there will be no ambiguity from the context. Since $BV[Q]\subset L^2[Q]$, for any $f,g\in BV[Q]$ we shall also use the notation $\langle f,g\rangle$ and $\|f\|$ for the inner product of f and g, and the norm of f in the sense of $L^2[Q]$ space respectively.

PROPOSITION 2. 2. *For $f, g \in BV[Q]$ and $\alpha, \beta \in R^1$, the stochastic integral I satisfies the following*:

(1) $I(f) \sim N(0, \|f\|^2)$.

(2) $\|I(f)\|^2 = \|f\|^2$.

(3) *The sequence* $\{S(f, P_n, \xi_n)\}$ *converges in* $L^2[C_2[Q]]$ *to* $I(f) \in L^2[C_2[Q]]$.

(4) $\langle I(f), I(g) \rangle = \langle f, g \rangle$.

(5) $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$.

Proof. (1) Since $S(f, P_n, \xi_n)$ is a linear combination of normally distributed random variables $Y((s,t), \cdot)$ on $C_2[Q]$, it is normally distributed and by Lemma 2. 1, we have $S(f, P_n, \xi_n) \sim N(0, \sum_{j=1}^{q(n)} \sum_{i=1}^{p(n)} |f(u_{n,i}, v_{n,j})|^2 (s_{n,i} - s_{n,i-1})(t_{n,j} - t_{n,j-1}))$. Thus the characteristic function of $S(f, P_n, \xi_n)$, $\psi_n(t)$ is given by $\psi_n(t) = \exp\left\{-\frac{t^2}{2} \sum_{j=1}^{q(n)} \sum_{i=1}^{p(n)} |f(u_{n,i}, v_{n,j})|^2 (s_{n,i} - s_{n,i-1})(t_{n,j} - t_{n,j-1})\right\}$ for every $t \in R^1$. Hence we have, every $t \in R^1$,

$$\lim_{n\to\infty} \psi_n(t) = \exp\left\{-\frac{t^2}{2}\|f\|^2\right\} \equiv \psi(t).$$

Since $S(f, P_n, \xi_n)(x)$ converges to $I(f)(x)$ for every $x \in C_2[Q]$, it follows from Levy's continuity Theorem [2; p. 332] that $\psi(t)$ is the characteristic function of $I(f)$, so that we have $I(f) \sim N(0, \|f\|^2)$.

(2) Since $E^y(I(f)) = 0$, the variance of $I(f)$ is given by $E^y(I(f)^2)$, so that we have $\|I(f)\|^2 = \|f\|^2$.

(3) From the definition of Riemann integral of f, we have

$$\begin{aligned}\lim_{n\to\infty} \|S(f, P_n, \xi_n)\|^2 &= \lim_{n\to\infty} \left\{\sum_{j=1}^{q(n)} \sum_{i=1}^{p(n)} |f(u_{n,i}, v_{n,j})|^2 (s_{n,i} - s_{n,i-1})(t_{n,j} - t_{n,j-1})\right\} \\ &= \int_Q |f(s,t)|^2 ds dt = \|f\|^2.\end{aligned}$$

Since $\lim_{n\to\infty} S(f, P_n, \xi_n)(x) = I(f)(x)$ for every $x \in C_2[Q]$, it follows from [6; p. 118] that $S(f, P_n, \xi_n)$ converges in $L^2[C_2[Q]]$ to $I(f)$.

(4) By using the result of (3), we have

$$\begin{aligned}\langle I(f), I(g) \rangle &= \lim_{n\to\infty} \langle S(f, P_n, \xi_n), S(g, P_n, \xi_n) \rangle \\ &= \lim_{n\to\infty} \left\{\sum_{j=1}^{q(n)} \sum_{i=1}^{p(n)} f(u_{n,i}, v_{n,j}) g(u_{n,i}, v_{n,j})(s_{n,i} - s_{n,i-1})(t_{n,j} - t_{n,j-1})\right\} \\ &= \int_Q f(s,t) g(s,t) ds dt = \langle f, g \rangle.\end{aligned}$$

(5) The proof follows from the fact

$$S(\alpha f + \beta g, P_n, \xi_n) = \alpha S(f, P_n, \xi_n) + \beta S(g, P_n, \xi_n).$$

REMARK 2.2. (a) Let $\{e_k | k=1, 2, \ldots\} \subset BV[Q]$ be a complete orthonormal set (*C. O. N.* set) for $L^2[Q]$. For $f \in L^2[Q]$, let $f_n(s, t) = \sum_{k=1}^{n} a_k e_k(s, t)$ where $a_k = \langle f, e_k \rangle$. Then since $\{f_n\}$ converges in $L^2[Q]$ to f, we note from (2) of Proposition 2.2 that there exists the element $\hat{I}(f)$ in $L^2[C_2[Q]]$ to which $\{I(f_n)\}$ converges in $L^2[C_2[Q]]$. We also note from (2) of Proposition 2.2 that the element $\hat{I}(f)$ in $L^2[C_2[Q]]$ is determined by f independently of the choice of the C. O. N. set $\{e_k\}$ for $L^2[Q]$.

(b) Let $\{e_k\}$ and $\{a_k\}$ be as in (*a*). Since $I(a_i e_i) \sim N(0, a_i^2)$ and covariance of $I(a_i e_i)$ and $I(a_j e_j)$ is zero for all i, j with $i \neq j$, $\{I(a_i e_i)\}$ is a sequence of independent random variables on $C_2[Q]$ with $\sum_{i=1}^{\infty} E^y(I(a_i e_i)^2) = \sum_{i=1}^{\infty} a_i^2 = \|f\|^2 < \infty$. Hence it follows from [3; p. 197] that $\sum_{i=1}^{\infty} I(a_i e_i)(x) = \lim_{n\to\infty} I(f_n)(x)$ exists for m_y-a. e. x in $C_2[Q]$.

(c) We note from (a) and (b) that for $f \in L^2[Q]$, $\hat{I}(f)(x) = \lim_{n\to\infty} I(f_n)(x)$ for m_y-a. e. x in $C_2[Q]$ and that for $f \in BV[Q]$, $\hat{I}(f)(x) = I(f)(x)$ for m_y-a. e. x in $C_2[Q]$.

DEFINITION 2.3. For $f \in L^2[Q]$, we define *the stochastic integral of f with respect to the Yeh-Wiener process* Y to be the element $I(f)$ in $L^2[C_2[Q]]$ of Remark 2.2.

The *P. W. Z. integral of f over Q with respect to* $x \in C_2[Q]$ is defined to be the real number $\hat{I}(f)(x)$ and will be denoted by

$$\hat{I}(f)(x) = \int_Q f \hat{d}x, \text{ or } \int_Q f(s, t)\ \hat{d}x(s, t).$$

PROPOSITION 2.3. *For $f, g \in L^2[Q]$ and $\alpha, \beta \in R^1$, the stochastic integral $\hat{I}$ satisfies* (1), (2), (4) *and* (5) *of* Proposition 2.2.

Proof. To prove (1) let $\{f_n\}$ and $\{a_n\}$ be as in Remark 2.2. Since $I(f_n) \sim N(0, \|f_n\|^2) = N\left(0, \sum_{k=1}^{n} a_k^2\right)$, by using the same arguement as in the proof of (a) in Proposition 2.2, we have $\hat{I}(f) \sim N\left(0, \sum_{k=1}^{\infty} a_k^2\right) = N(0, \|f\|^2)$.

The proofs of (2), (4) and (5) are similar to the corresponding proofs of (2), (4) and (5) in Proposition 2.2.

The following theorem is a slight extension of the *P. W. Z.* Theorem for $C_2[Q]$ in [12; p. 1430]. It may be noted that our proof of this theorem is rather simpler than that of Yeh [12].

THEOREM 2.4. *Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an orthogonal set in $L^2[Q]$, and let $f(u_1, \ldots, u_n)$ be a real or complex valued Lebesgue measurable function on R^n. Then the functional F on $C_2[Q]$ defined by*

$$F(x)=f(\textstyle\int_Q\alpha_1 \hat{d}x, \ldots, \int_Q\alpha_n \hat{d}x), \quad x\in C_2[Q] \tag{2.1}$$

is Yeh-Wiener measurable and

$$E^y(F) \stackrel{*}{=} \{(2\pi)^n \prod_{j=1}^n \|\alpha_j\|^2\}^{-\frac{1}{2}} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f(u_1, \ldots, u_n) \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{u_j^2}{\|\alpha_j\|^2}\right\} du_1 \ldots du_n \tag{2.2}$$

where $\stackrel{*}{=}$ means that the existence of one side implies that of the other with the equality.

Proof. Let $\hat{I}_n(x)=(\hat{I}(\alpha_1)(x), \ldots, \hat{I}(\alpha_n)(x))$, $x\in C_2[Q]$. Then $\hat{I}_n$ is an n-dimensional random vector on $C_2[Q]$. Since $\hat{I}(\alpha_i) \sim N(0, \|\alpha_i\|^2)$ and $\langle \hat{I}(\alpha_i), \hat{I}(\alpha_j)\rangle = \langle \alpha_i, \alpha_j\rangle = 0$ for all i, j with $i \neq j$, it follows that $\{\hat{I}(\alpha_i) \mid i=1, 2, \ldots, n\}$ is a set of independent random variables, so that the probability distribution of $\hat{I}_n$ on R^n, $m_y \circ \hat{I}_n^{-1}$ is equal to $m_y \circ \hat{I}^{-1}(\alpha_1) \times \ldots \times m_y \circ \hat{I}^{-1}(\alpha_n)$. Hence we have, for every Borel set B in R^n

$$m_y(\hat{I}_n^{-1}(B)) = \left\{(2\pi)^n \prod_{j=1}^n \|\alpha_j\|^2\right\}^{-\frac{1}{2}} \cdot \int(n) \int_B \exp\left\{-\frac{1}{2}\sum_{j=1}^n \frac{u_j^2}{\|\alpha_j\|^2}\right\} du_1 \ldots du_n. \tag{2.3}$$

Now let us show that $\hat{I}_n$ is a measurable transformation of $(C_2[Q], \mathcal{Y}, m_y)$ into $(R^n, \mathcal{M})$ where $\mathcal{M}$ denotes the σ-algebra of all Lebesgue measurable subsets of R^n. To show this we need to show that $\hat{I}_n^{-1}(E) \in \mathcal{Y}$ for every $E \in \mathcal{M}$. Now let $E \in \mathcal{M}$ and $E = B \cup N_1$, where B is a Borel set in R^n and N_1 is a subset of a null Borel set N in R^n. Then $\hat{I}_n^{-1}(B)$ and $\hat{I}_n^{-1}(N)$ are in $\mathcal{Y}$. Since m_y is complete and $m_y(\hat{I}_n^{-1}(N_1))=0$ by (2.3), we have $\hat{I}_n^{-1}(E) \in \mathcal{Y}$. Hence if f is a Lebesgue measurable function on R^n, then $F=f\circ\hat{I}_n$ defined by (2.1) is a Yeh-Wiener measurable functional on $C_2[Q]$.

The formular in (2.2) can be easily obtained by using the change of variables theorem [3; p. 163] and (2.3).

3. Conditional Yeh-Wiener Integral

Lex X be a real valued Yeh-Wiener measurable functional on $(C_2[Q], \mathcal{Y}, m_y)$. The probability distribution of X is by definition a probability measure P_x on $(R^1, \mathcal{B}(R^1))$ given by

$$P_x(B)=m_y(X^{-1}(B)) \text{ for } B\in\mathcal{B}(R^1),$$

where $\mathcal{B}(R^1)$ denotes the σ-algebra of the Borel sets in R^1.

DEFINITION 3.1. Let X and Z be real valued Yeh-Wiener measurable functionals on $(C_2[Q].\mathcal{Y}, m_y)$ with $E^y[|Z|]<\infty$. The conditional Yeh-Wiener integral of Z given X, written $E^y(Z|X)$, is defined to be the equivalence class of $\mathcal{B}(R^1)$-measurable and P_x-integrable functions f on R^1 modulo P_x-a. e. equality on R^1 such that for $B\in\mathcal{B}(R^1)$,

$$\int_{X^{-1}(B)} Z(x)dm_y(x)=\int_B f(w)dP_x(w). \tag{3.1}$$

REMARK 3.1 By Radon-Nikodym Theorem a function f on R^1 satisfying (3.1) exists, and if g is another such function on R^1, then $f(w)=g(w)$ for P_x-a. e. $w\in R^1$. We shall also use $E^y(Z|X)$ to mean a particular version of the equivalence class. Thus the equation (3.1) can be written as

$$\int_{X^{-1}(B)} Z(x)dm_y(x)=\int_B E^y(Z|X)(w)dP_x(w). \tag{3.2}$$

The following results (Lemma 3.1, Corallary 3.2, Theorems 3.3-3.5) are simple modifications of the corresponding results for the Wiener space in [10]. The proofs in our setting remain the same.

LEMMA 3.1 *Let X and Z be real valued Yeh-Wiener measurable functionals on $C_2[Q]$ with $E^y[|Z|]<\infty$. Then for any $\mathcal{B}(R^1)$-measurable function g on R^1, we have*

$$E^y[(g\cdot X)Z] \overset{*}{=} \int_{R^1} g(w)E^y(Z|X)(w)dP_x(w).$$

CORALLARY 3.2. *Let X and Z be as in Lemma 3.1. Assume that P_x is absolutely continuous with respect to Lebesgue measure m on $(R^1, \mathcal{B}(R^1))$. For every $w\in R^1$ and $a>0$, let J_a^w be the function on R^1 defined by*

$$J_a^w(\xi)=\begin{cases}\dfrac{1}{2a}, & \xi\in[w-a,\ w+a]\\ 0, & \xi\notin[w-a,\ w+a].\end{cases}$$

Then there exists a version of $E^y(Z|X)\dfrac{dP_x}{dm}$ such that for $w\in R^1$,

$$E^y(Z|X)(w)\frac{dP_x}{dm}(w)=\lim_{a\to 0} E^y[(J_a^w\cdot X)Z] \tag{3.3}$$

where $\frac{dP_x}{dm}$ *denotes the Radon–Nikodym derivative of* P_x *with respect to* m.

THEOREM 3.3. *Let* X *and* Z *be real valued Yeh–Wiener measurable functionals on* $C_2[Q]$ *with* $E^y[|Z|]<\infty$. *Assume that* P_x *is absolutely continuous with respect to* m *on* $(R^1, \mathscr{B}(R^1))$. *Then there exists a version of* $E^y(Z|X)\frac{dP_x}{dm}$ *such that for* $w\in R^1$,

$$E^y(Z|X)(w)\frac{dP_x}{dm}(w)=\lim_{h\to\infty}\frac{1}{2\pi}\int_{(-h,h)}\left(1-\frac{|u|}{h}\right)e^{-iuw}E^y[e^{iuX}Z]dm(u).$$

THEOREM 3.4. *Let* X *and* Z *be as in* Theorem 3.3. *For* a, $b\in R^1$ *with* $a<b$, *let* J_a^b *be the function on* R^1 *defined by*

$$J_a^b(w)=\begin{cases}1, & w\in(a,b)\\ 0, & w\notin[a,b]\\ \frac{1}{2}, & w=a,\ w=b.\end{cases}$$

Then we have

$$\int_{R^1}J_a^b(w)\ E^y(Z|X)(w)\frac{dP_x}{dm}(w)\ dm(w)$$
$$=\lim_{h\to\infty}\frac{1}{2\pi}\int_{(-h,h)}\frac{e^{-iau}-e^{ibu}}{iu}E^y[e^{iuX}Z]\ dm(u).$$

THEOREM 3.5. *Let* X *and* Z *be as in* Theorem 3.3. *Assume that* $E^y(e^{iuX}Z)$ *is a Lebesgue integrable function of* u *on* $(R^1, \mathscr{B}(R^1))$. *Then there exists a version of* $E^y(Z|X)\frac{dP_x}{dm}$ *such that for* $w\in R^1$,

$$E^y(Z|X)(w)\frac{dP_x}{dm}(w)=\frac{1}{2\pi}\int_{R^1}e^{-iuw}E^y[e^{iuX}Z]dm(u).$$

As examples of evaluation of conditional Yeh–Wiener integrals we have the following:

EXAMPLE 1. Let X and Z be real valued Yeh–Wiener measurable functionals on $C_2[Q]$ defined by, respectively,

$$X(x)=x(p,q)\ \text{ and }\ Z(x)=\frac{1}{pq}\int_0^q\int_0^p x(u,v)\,du\,dv\ \text{ for } x\in C_2[Q].$$

Then Z is Yeh–Wiener integrable since it can be shown that $E^y[|Z|]=\frac{4\sqrt{2}}{6}\sqrt{\frac{pq}{\pi}}<\infty$ by applying the Fubini theorem. Hence $E^y(Z|X)$ exists.

Now let us find $E^y(Z|X)(w)$, $w\in R^1$. We first note that from the fact that $Y((p,q),\cdot)\sim N(0,pq)$, we have

$$\text{(3.4)}\qquad \frac{dP_x}{dm}(w)=\frac{1}{\sqrt{2\pi pq}}\exp\left(-\frac{w^2}{2pq}\right)\ \text{for}\ w\in R^1.$$

By Corollary 3.2, there exists a version of $E^y(Z|X)\frac{dP_x}{dm}$ such that

$$\text{(3.5)}\qquad E^y(Z|X)(w)\frac{dP_x}{dm}(w)=\lim_{a\to 0} E^y[(J_a^w\cdot X)Z]\ \text{for}\ w\in R^1.$$

But with our X and Z, we have

$$\text{(3.6)}\qquad E^y[J_a^w\cdot X)Z]=E^y[J_a^w(x(p,q))\frac{1}{pq}\int_0^q\int_0^p x(u,v)\,du\,dv].$$

We apply the Fubini Theorem to the right-hand side of (3.6) to obtain

$$\text{(3.7)}\qquad E^y[(J_a^w\cdot X)Z]=\frac{1}{pq}\int_0^q\int_0^p E^y[J_a^w(x(p,q))x(u,v)]\,du\,dv.$$

Evaluating the integrand of the integral of the right-hand side of (3.7) with the long and tedious computations (see [1]), we obtain

$$\begin{aligned}\text{(3.8)}\qquad & E^y[J_a^w(x(p,q))x(u,v)]\\ &=\frac{1}{2a}\frac{1}{(2\pi)^2uv(p-u)(q-v)}\int_{w-a}^{w+a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_{11}\cdot\\ &\quad\exp\left\{-\frac{u_{11}^2}{2uv}-\frac{(u_{12}-u_{11})^2}{2u(q-v)}-\frac{(u_{21}-u_{11})^2}{2v(p-u)}\right.\\ &\quad\left.-\frac{(u_{22}-u_{12}-u_{21}+u_{11})^2}{2(p-u)(q-v)}\right\}du_{21}du_{12}du_{11}du_{22}\\ &=\frac{1}{2apq}\frac{uv}{2\pi pq}\int_{w-a}^{w+a}u_{22}\exp\left(-\frac{u_{22}}{2pq}\right)du_{22}.\end{aligned}$$

Substituting (3.8) for the right-hand side of (3.7), we obtain

$$\text{(3.9)}\qquad E^y[(J_a^w\cdot X)Z]=\frac{1}{4}\frac{1}{\sqrt{2\pi pq}}\frac{1}{2a}\int_{w-a}^{w+a}u_{22}\exp\left(-\frac{u_{22}^2}{2pq}\right)du_{22}.$$

Combining (3.5) and (3.9), we have

$$\begin{aligned}E^y(Z|X)(w)\frac{dP_x}{dm}(w)&=\lim_{a\to 0}\left\{\frac{1}{4}\frac{1}{\sqrt{2\pi pq}}\frac{1}{2a}\int_{w-a}^{w+a}u_{22}\exp\left(-\frac{u_{22}^2}{2pq}\right)du_{22}\right\}\\ &=\frac{w}{4}\frac{1}{\sqrt{2\pi pq}}\exp\left(-\frac{w^2}{2pq}\right),\quad w\in R^1.\end{aligned}$$

Hence it follows from (3.4) that we obtain

$$E^y(Z|X)(w)=\frac{w}{4}\ \text{for}\ w\in R^1.$$

EXAMPLE 2. Let X and Z be real valued Yeh-Wiener measurable functionals on $C_2[Q]$ defined by, respectively,

$$X(x)=x(p,q) \text{ and } Z(x)=\int_0^q\int_0^p [x(u,v)]^2\, dudv \text{ for } x\in C_2[Q].$$

We first note that Z is Yeh-Wiener integrable and $E[Z]=\frac{(pq)^2}{4}$. Thus $E^y(Z|X)(w)$ exists. Let us find $E^y(Z|X)(w)$ in the way that we proceeded in Example 1. To apply Corollary 3.2, observe that

$$(3.10)\qquad \begin{aligned} E^y[J_a^w\cdot X)Z] &= E^y[J_a^w(x(p,q))\int_0^q\int_0^p [x(u,v)]^2 dudv] \\ &= \int_0^q\int_0^s E^y[J_a^w(x(p,q))[x(u,v)]^2]dudv. \end{aligned}$$

Evaluating of the integrand of the integral of the right-hand side of (3.10) with the long and tedious computations (See [1]), we obtain

$$(3.11)\qquad \begin{aligned} &E^y[J_a^w(x(p,q))[x(u,v)]^2] \\ &= \frac{1}{2a}\frac{1}{(2\pi)^2uv(p-u)(q-v)}\int_{w-a}^{w+a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{11}^2 \exp\left\{-\frac{u_{11}^2}{2uv}\right. \\ &\left.-\frac{(u_{12}-u_{11})^2}{2u(q-v)}-\frac{(u_{21}-u_{11})^2}{2v(p-u)}-\frac{(u_{22}-u_{12}-u_{21}+u_{11})^2}{2(p-u)(q-v)}\right\} \\ &du_{21}du_{12}du_{11}du_{22} \\ &= \frac{1}{2a}\frac{1}{\sqrt{2\pi pq}}\int_{w-a}^{w+a}\left[\frac{uv(pq-uv)}{pq}+\left(\frac{uv}{pq}\right)^2 u_{22}^2\right] \exp\left(-\frac{u_{22}^2}{2pq}\right)du_{22}. \end{aligned}$$

From (3.10) and (3.11), we have

$$\begin{aligned} &E^y[(J_a^w\cdot X)Z] \\ &= \frac{1}{2a}\frac{1}{\sqrt{2\pi pq}}\int_{w-a}^{w+a}\left\{\frac{5}{36}(pq)^2+\frac{pq}{9}u_{22}^2\right\} \exp\left(-\frac{u_{22}^2}{2pq}\right) du_{22}. \end{aligned}$$

Hence we have

$$(3.12)\qquad \begin{aligned} E^y(Z|X)(w)\frac{dP_x}{dm}(w) &= \lim_{a\to 0} E^y[(J_a^w\cdot X)Z] \\ &= \frac{1}{\sqrt{2\pi pq}}\left\{\frac{5}{36}(pq)^2+\frac{pq}{9}w^2\right\} \exp\left(-\frac{w^2}{2pq}\right). \end{aligned}$$

Since $\frac{dP_x}{dm}(w)=\frac{1}{\sqrt{2\pi pq}} \exp\left(-\frac{w^2}{2pq}\right)$, we obtain

$$E^y(Z|X)(w)=\frac{5}{36}(pq)^2+\frac{pq}{9}w^2,\quad w\in R^1.$$

It is interesting to note that from (3.1) and (3.12)

$$E^y[Z]=\int_{X^{-1}(R^1)} Z(x)dm_y(x)=\int_{R^1} E^y(Z|X)(w)\ dP_x(w)$$
$$=\frac{1}{\sqrt{2\pi pq}}\int_{R^1}\left[\frac{5}{36}(pq)^2+\frac{pq}{9}w^2\right]\ \exp\left(-\frac{w^2}{2pq}\right)\ dm(w)$$
$$=\frac{(pq)^2}{4}$$

as can be obtained by a direct computation of $E^y[Z]$.

The following theorem is an extension of Theorem 4 in [10; p. 663] for the Yeh-Wiener measure space $(C_2[Q], \mathcal{Y}, m_y)$.

THEOREM 3.6. *Let X and Z be two Yeh-Wiener functionals on $C_2[Q]$ defined by, respectively*

$$X(x)=x(p, q) \text{ and}$$
$$Z(x)=f\left(\int_Q \alpha_1(s, t)\hat{d}x(s, t),\ \dots,\ \int_Q \alpha_n(s, t)\hat{d}x(s, t)\right),\quad x\in C_2[Q]$$

where $\{\alpha_1, \alpha_2, \dots, \alpha_n, 1\}$ is an orthogonal set in $L^2[Q]$, and f is a real valued Lebesgue measurable function on R^n such that

$$(3.13)\qquad \{(2\pi)^n\prod_{j=1}^n \|\alpha_j\|^2\}^{-\frac{1}{2}}$$
$$\cdot\int_{R^n}|f(u_1, \dots, u_n)|\exp\left(-\frac{1}{2}\sum_{j=1}^n\frac{u_j^2}{\|\alpha_j\|^2}\right)\ du_1\dots du_n<\infty.$$

Then we have the following:

(1) *Z is a real valued Yeh-Wiener integrable on functional $C_2[Q]$.*

(2) *There exists a version of $E^y\ (Z|X)$ such that $E^y\ (Z|X)(w)=M$ for $w\in R^1$ where $M=E^y[Z]$ given by (2.2).*

Proof. (1) From (3.13) and (2.2), we have $E^y[|Z|]<\infty$, and hence Z is real valued Yeh-Wiener integrable functional on $C_2[Q]$ and its integral is given by the right-hand side of (2.2).

(2) Since $\int_Q 1\ \hat{d}x=x(p, q)=X(x)$ for m_y-a. e. $x\in C_2[Q]$, we have

$$E^y[e^{iuX}Z]=E^y\left[\exp\left(iu\int_Q 1\hat{d}x\right)f\left(\int_Q\alpha_1\hat{d}x,\ \dots,\ \int_Q\alpha_n\hat{d}x\right)\right].$$

Applying (2.2) to the function h defined by $h(u_1, \dots, u_n, w)e^{iuw}f(u_1, \dots, u_n)$ for $(u_1, \dots, u_n,\ w)\in R^{n+1}$, we have

$$E^y[e^{iuX}Z]=M\cdot\frac{1}{\sqrt{2\pi pq}}\int_{-\infty}^{\infty}\exp(iuw)\ \exp\left(-\frac{w^2}{2pq}\right)\ dw$$

$$=M\cdot\exp\left(-\frac{pqu^2}{2}\right),\quad u\in R^1.$$

Since $\int_{R^1}|E^y[e^{iuX}Z]|\,du<\infty$, by Theorem 3.5 it follows that there exists a version of $E^y(Z|X)\frac{dP_x}{dm}$ such that

$$E^y(Z|X)(w)\frac{dP_x}{dm}(w)=\frac{M}{2\pi}\int_{-\infty}^{\infty}\exp(-iuw)\ \exp\left(-\frac{pqu^2}{2}\right)du$$
$$=\frac{M}{\sqrt{2\pi pq}}\exp\left(-\frac{w^2}{2pq}\right).$$

Hence by (3.4), we have

$$E^y(Z|X)(w)=M \text{ for } w\in R^1.$$

As examples of application of Theorem 3.6 in evaluating conditional Yeh–Wiener integrals we have the following:

EXAMPLE 3. Let X and Z be Yeh–Wiener functionals on $C_2[Q]$ defined by, respectively

$$X(x)=x(p,q) \text{ and } Z(x)=f\left(\int_Q\alpha\hat{d}x\right) \text{ for } x\in C_2[Q],$$

where $\{\alpha_1, 1\}$ is an orthogonal set in $L^2[Q]$ and f is a function on R^1 defined by $f(u)=u^n$ where n is a natural number. Then we have

$$\frac{1}{\sqrt{2\pi}\|\alpha\|}\int_{R^1}|u|^n\exp\left\{-\frac{1}{2}\frac{u^2}{\|\alpha\|^2}\right\}du=\left\{\frac{2^n}{\pi}\right\}^{\frac{1}{2}}\|\alpha\|^n\Gamma\left(\frac{n+1}{2}\right)<\infty$$

where Γ is the gamma function. Hence $E^y[|Z|]<\infty$; i.e. Z is Yeh–Wiener integrable and its integral $E^y[Z]$ is given by

$$E^y[Z]=M_n=\begin{cases}0\quad, & \text{if } n=2k-1\\ 1\cdot3\cdots(2k-1)\|\alpha\|^{2k}, & \text{if } n=2k\end{cases}$$

Thus we have a version of $E^y(Z|X)$ such that $E^y(Z|X)(w)=M_n$ for $w\in R^1$.

EXAMPLE 4. Let X and Z be two functionals on $C_2[Q]$ as in Example 3 with f on R^1 defined by $f(u)=\exp(\xi u)$, where $\xi\in R^1$. Then we have

$$\frac{1}{\sqrt{2\pi}\|\alpha\|}\int_{R^1}|\exp(\xi u)|\exp\left\{-\frac{1}{2}\frac{u^2}{\|\alpha\|^2}\right\}du=\exp\left\{\frac{\xi^2}{2}\|\alpha\|^2\right\}<\infty.$$

Hence $E^y[|Z|]<\infty$ and $E^y[Z]$ is given by $\exp\left\{\frac{\xi^2}{2}\|\alpha\|^2\right\}$. Thus we have a version of $E^y(Z|X)$ such that $E^y(Z|X)(w)=\exp\left\{\frac{\xi^2}{2}\|\alpha\|^2\right\}$ for $w\in R^1$.

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