J. Korean Math. Soc.
Vol. 20, No. 2, 1983

INTEGRAL REPRESENTATIONS OF SOLUTIONS OF CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

Throughout this paper Ω will stand for an open neighborhood $\{(x,t)\mid |x|<r,\ |t|<\delta\}$ of the origin of R^2. We consider the general linear partial differential operator L of the first order in Ω with the canonical form

$$L=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}$$

where $b(x,t)$ is a real valued C^∞ function in Ω.

The linear partial differential operator L is said to satisfy *condition* (P_0) in Ω if

(i) for any $x\in(-r,r)$ the function $t\to b(x,t)$ does not change sign in the interval $|t|<\delta$,

in such a way that

(ii) $b(0,0)=0$, and

(iii) for any $(x,t)\in\Omega\setminus\{(0,0)\}$ $b(x,t)\neq 0$,

and to satisfy *condition* (P_1) if

(i) for any $x\in(-r,r)$ the function $t\to b(x,t)$ does not change sign in the interval $|t|<\delta$,

in such a way that

(ii) $b(0,t)=0$ for any t, $|t|<\delta$, and

(iii) $b(x,t)\neq 0$ for any $(x,t)\in\Omega$ with $x\neq 0$.

We say that L belongs to $F(\Omega)$, a class of linear partial differential operators of the first order in Ω, if the Cauchy problem

$$Lz=\frac{\partial z}{\partial t}+ib(x,t)\frac{\partial z}{\partial x}=0$$

$$z|_{t=0}=x$$

has a C^∞ solution in an open neighborhood $U(U\subset\Omega)$ of the origin. Thus

Received September 10, 1983
This research is partially supported by the research grant of the Korean Science and Engineering Foundation, 1983.

if $b(x,t)$ is real analytic, then L belongs to $F(\Omega)$.

We now assume that L belongs to $F(\Omega)$ and satisfies (P_0) or (P_1). The principal aim of this paper is to represent a C^∞ solution of $Lu=f$ in an integral form in a neighborhood of the origin when $f\in C_0^\infty(\Omega)$. That $Lu=f$ is locally solvable follows immediately from the general criteria for the local solvability of linear partial differential operators due to Nirenberg–Treves [1]. When $b(x,t)$ is real analytic, the integral representation of a C^∞ solution of $Lu=f$ is established in Treves [2]. Thus our results partially generalize those of Treves [2] but in a way different from the Treves' arguments in [3] for linear partial differential equations of the first order with C^∞ coefficients.

2. A lemma

Let L be an element in $F(\Omega)$ and $z=z(x,t)$ be the C^∞ solution of the Cauchy problem

(1)
$$\frac{\partial z}{\partial t}+ib(x,t)\frac{\partial z}{\partial x}=0, \quad z|_{t=0}=x.$$

in a neighborhood of the origin

We write $z(x,t)=\xi(x,t)+i\eta(x,t)$ where ξ and η are real valued. From the initial condition (1) we know that $\frac{\partial \xi}{\partial x}\neq 0$ when $t=0$. Therefore we have the right to change variables

(2)
$$y=\xi(x,t), \quad s=t$$

in $U_0(U_0\subset U)$, an open neighborhood of the origin in R^2. Let

(3)
$$z=y+i\phi(y,s)$$

where $\phi(y,s)=\eta(x,t)$ real valued, C^∞ and

(4)
$$\phi(y,0)=0.$$

In (y,s) coordinates we have

$$\begin{aligned} L&=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x} \\ &=\frac{\partial}{\partial s}+\left\{\frac{\partial y}{\partial t}+ib(x,t)\frac{\partial y}{\partial x}\right\}\frac{\partial}{\partial y} \\ &=\frac{\partial}{\partial s}+\lambda(y,s)\frac{\partial}{\partial y} \end{aligned}$$

But $Lz=0$, that is,

$$0=L(y+i\phi)=i\phi_s+\lambda(1+i\phi_y).$$

We derive from this

$$\lambda=\frac{\partial y}{\partial t}+ib\frac{\partial y}{\partial x}=-i\phi_s/(1+i\phi_y).$$

Since y is a real valued function we obtain

$$b(x,t)=-(1+\phi_y^2)^{-1}\left(\frac{\partial y}{\partial x}\right)^{-1}\phi_s(y,s). \tag{5}$$

From the above equality we can prove the following important lemma.

LEMMA. *Let L be an element in $F(\Omega)$ satisfying either (P_0) or (P_1). Let the C^∞ map $\phi: (x,t)\to(y,s)$ be the local coordinate change in $U_0\subset\Omega$ as is defined in* (2) *and $\bar{r}$, $\bar{\delta}$ be the positive real numbers such that*

$$\{(y,s)\,|\,|y|<\bar{r},\ \ |s|<\bar{\delta}\}\subset\phi(U_0).$$

Then for every $y\in(-\bar{r},\bar{r})$, the function $s\to\phi_s(y,s)$ does not change sign in the interval $\{s\,|\,-\bar{\delta}<s<\bar{\delta}\}$.

Proof. From (5) we know that ϕ_s is nonzero as soon as $b(x,t)$ is nonzero. We also note that $b(x,t)>0$ (or $b(x,t)<0$) for all $(x,t)\in\Omega\setminus\{(0,0)\}$ if L satifies (P_0) and that $b(x,t)$ is nonzero and does not change sign for $x>0$ (or $x<0$) if L satisfies (P_1).

Thus in the case when L satisfies (P_0) it follows immediately from (5) that for fixed $y\in(-\bar{r},\bar{r})$ $\phi_s(y,s)$ does not change sign in the interval $|s|<\bar{\delta}$.

Now assume that L satisfies (P_1). We claim that ϕ maps $\{(0,t)\,|\,(0,t)\in U_0\}$ into $\{(0,s)\,|\,|s|<\delta\}$ in (y,s) plane. In fact, from the condition (P_1) $b(0,t)=0$ for any t, $|t|<\delta$. Therefore, $L=\frac{\partial}{\partial t}$ when $x=0$. Thus, on $\{(0,t)\,|\,(0,t)\in U_0\}$ $\frac{\partial z}{\partial t}=0$, that, is, $\frac{\partial}{\partial t}(\xi+i\eta)=0$. Hence $\frac{\partial\xi}{\partial t}(0,t)=0$ for any t, $|t|<\delta$. This shows that

$$y(0,t)\equiv\xi(0,t)=\text{const for any } t,\ \ |t|<\delta.$$

On the other hand, $z|_{t=0}\equiv\xi(x,0)=x$. Hence, in particular, $\xi(0,0)=0$. Therefore

$$y(0,\mathrm{t})\equiv\xi(0,t)=0 \text{ for any } t,\ \ |t|<\delta.$$

Thus ϕ maps $\{(0,t)\,|\,(0,t)\in U_0\}$ into $\{(0,s)\,|\,|s|<\delta\}$.

From the above fact and the equality (5), it follows that $\phi_s(0,s)=0$ for any s, $|s|<\bar{\delta}$ and hence $s\to\phi_s(0, s)$ does not change sign in the interval $|s|<\bar{\delta}$.

Now since ϕ is a bijective map, the inverse image of $\{(y, s)\mid |s|<\bar{\delta}\}$ for fixed $y\neq 0$ under ϕ does not intersect with $\{(0, t)\mid (0, t)\in U_0\}$ and is entirely contained either in $\{(x, t)\in U_0\mid x>0\}$ or in $\{(x, t)\in U_0\mid x<0\}$.

Therefore from the equality (5) together with the condition (P_1) for fixed $y\in(-\bar{r}, \bar{r})$, $y\neq 0$ the function $s\to\phi_s(y, s)$ does not change sign in the interval $|s|<\bar{\delta}$. Q. E. D.

3. Integral representations of solutions

We first consider the case when L satisfies the condition (P_1). In this case, note that for any fixed $y\neq 0$, $|y|<\bar{r}$ we have

$$\phi_s(y, s)>0 \text{ (or } \phi_s(y, s)<0) \text{ for all } s, \ |s|<\bar{\delta}.$$

This implies that if $y\neq 0$ and $|y|<\bar{r}$, then the map $s\to\phi(y, s)$ is a strictly increasing (or a strictly decreasing) function in the interval $|s|<\bar{\delta}$.

We subdivide the open rectangle

$$|y|<\bar{r}, \quad |s|<\bar{\delta} \tag{6}$$

as a union of

$$\mathscr{L}=\{(0, s)\mid \ |s|<\bar{\delta}\}$$

and open rectangles

$$R^+=\{(y, s)\mid \ 0<y<\bar{r}, \ |s|<\bar{\delta}\}$$

and

$$R^-=\{(y, s)\mid \ -\bar{r}<y<0, \ |s|<\bar{\delta}\}.$$

We note that the ranges of the map $z=y+i\phi(y, s)$ restricted to the rectangle (6) are as follows:

(i) z maps $\mathscr{L}$ to the single point 0 and

(ii) z maps the rectangles R^+ and R^- homeomorphically onto open sets θ_1 and θ_2 of complex plane C which contain, respectively, the real intervals

$$0<\operatorname{Re} z<\bar{r}, \ \operatorname{Im} z=0$$

and

$$-\bar{r}<\operatorname{Re} z<0, \ \operatorname{Im} z=0$$

and which are entirely contained, respectively, in the strip

$$0<\operatorname{Re} z<\bar{r}$$

and in the strip

$$-\bar{\delta}<\operatorname{Re} z<0.$$

We shall denote by A the image of rectangle (6) under ϕ.

Let now $f(x,t)$ be any C^∞ function in R^2 with support contained in

$$V=\phi^{-1}\{(y,s)\,|\,|y|<\bar{r},\quad |s|<\bar{\delta}\}. \tag{7}$$

We note that the equation

$$Lu=\frac{\partial u}{\partial t}+ib(x,t)\frac{\partial u}{\partial x}=f \tag{8}$$

is equivalent to

$$\begin{aligned}\left(\frac{\partial}{\partial s}+\lambda\frac{\partial}{\partial y}\right)&\left(u(y,s)-\int_{-\bar{\delta}}^{s}f(y,\sigma)d\sigma\right)\\ &=-\lambda(y,s)\int_{-\bar{\delta}}^{s}\left(\frac{\partial f}{\partial y}\right)(y,\sigma)d\sigma.\end{aligned} \tag{9}$$

Here $f(y,s)=f(x(y,s),t)$ etc. For the simplicity of our argument we shall set

$$v(y,s)=u(y,s)-\int_{-\bar{\delta}}^{s}f(y,\sigma)d\sigma$$

$$g(y,s)=-\lambda(y,s)\int_{-\bar{\delta}}^{s}\left(\frac{\partial f}{\partial y}\right)(y,\sigma)d\sigma$$

As $\lambda=-i\phi_s/(1+i\phi_y)$ vanishes identically on the vertical line segment $\mathcal{L}$ (where $\phi_s=0$), we have $g=0$ on $\mathcal{L}$.

Now we transfer v and g to the set A under the map $z=y+i\phi(y,s)$. Since $g=0$ on $\mathcal{L}$ and since z is a homeomorphism on R^+ and R^-, the transferred function $\tilde{g}(z)$ can be extended by 0 outside of A and is equal to a compactly supported function of L^1 class, with compact support contained in $\bar{A}$. The equation (9) becomes

$$\left(\frac{\partial\bar{z}}{\partial s}+\lambda(y,s)\frac{\partial\bar{z}}{\partial y}\right)\left(\frac{\partial\tilde{v}}{\partial\bar{z}}\right)=\tilde{g} \tag{10}$$

where $\tilde{v}$ denote $v(y,s)$ as a function of z.

But since $\lambda\left(\frac{\partial z}{\partial y}\right)=-\frac{\partial z}{\partial s}$, we have $\frac{\partial\bar{z}}{\partial s}=-\bar{\lambda}\left(\frac{\partial\bar{z}}{\partial y}\right)$. Therefore (10) reads

$$2i(Im\lambda)^{\sim}\left(\frac{\partial\bar{z}}{\partial y}\right)\left(\frac{\partial\tilde{v}}{\partial\bar{z}}\right)=\tilde{g}. \tag{11}$$

Moreover, since $\frac{\partial\bar{z}}{\partial y}=1-i\phi_y$ and $\operatorname{Im}\lambda=-\phi_s/(1+\phi_y^2)$, (11) is equivalent to

$$[\{-2i/(1+i\phi_y)\}\phi_s]^{\sim}\frac{\partial\tilde{v}}{\partial\bar{z}}=\tilde{g} \tag{12}$$

or,

$$\frac{\partial\tilde{v}}{\partial\bar{z}}=\frac{i}{2}[(1+i\phi_y)g/\phi_s]^{\sim}=-\frac{1}{2}\int_{-\bar{\delta}}^{s}\left(\frac{\partial f}{\partial y}\right)(y,\sigma)d\sigma. \tag{13}$$

(13) is an inhomogeneous Cauchy Riemann equation whose solution is given by

$$\tilde{v}=\frac{1}{2\pi i}\iint\frac{F(\zeta)}{z-\zeta}d\bar{\zeta}\Lambda d\zeta \tag{14}$$

where

$$F(z)=\frac{i}{2}[(1+i\phi_y)g/\phi_s]^{\sim}=\left[-\frac{1}{2}\int_{-\bar{\delta}}^{s}\left(\frac{\partial f}{\partial y}\right)(y,\sigma)\alpha\sigma\right]^{\sim}$$

To revert (14) to (y, s) coordinates, we set

$$\zeta=y'+i\phi(y', s').$$

Then we have

$$d\bar{\zeta}\,\Lambda\,d\zeta=2i\phi_{s'}dy'\,\Lambda\,ds'$$

and hence

$$v(y,s)=\frac{-1}{2\pi}\iint_{R^2}\frac{\phi_s(y',s')k(y',s')}{y-y'+i(\phi(y,s)-\phi(y',s'))}dy'ds'. \tag{15}$$

where

$$k(y,s)=\int_{-\bar{\delta}}^{s}\left(\frac{\partial f}{\partial y}\right)(y,\sigma)d\sigma$$

Since $v(y, s)$ is the pull back via

$$(y,s)\to y+i\phi(y,s)$$

of $\tilde{v}$ which is locally L^1 function, $v(y, s)$ is well defined and is, in fact, a C^∞ function.

So far we considered only the case when L satisfies the condition (P_1). When L satisfies (P_0), the argument is much simpler, as z is a homeomorphism on the entire rectangle (6) in this case. Thus we get the representation (15) even if L satisfies (P_0). Summing up, we have

THEOREM. *Let L be an element in $F(\Omega)$ satisfying either (P_0) or (P_1). Let V be an open neighborhood of the origin as is defined in* (7). *Then for any $f\in C_0^\infty(V)$ a C^∞ solution of the linear partial differential equation*

$$Lu=f$$

in V is given by the pull back via the map

$$\phi : (x, t) \to (y, s)$$

defined in (2) *of the* C^∞ *function*

$$u(y, s) = \frac{-1}{2\pi} \iint_{R^2} \frac{\phi_s(y', s') k(y', s')}{y - y' + i(\phi(y, s) - \phi(y', s'))} dy' ds' + \int_{-\bar{\delta}}^{s} f(y, \sigma) d\sigma$$

where

$$k(y, s) = \int_{-\bar{\delta}}^{s} \left(\frac{\partial f}{\partial y}\right)(y, \sigma) d\sigma$$

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