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THE INVARIANT DISTANCE DEFINED BY POSITIVE PLURIHARMONIC FUNCTIONS

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In this paper we study the pseudodistance defined by positive pluriharmonic functions.

Let M be a complex manifold. We call a real valued function u on M harmonic (=pluriharmonic) if, it is given locally by the real part of a holomorphic function. That is, for each $z_0 \in M$, there is a holomorphic function defined on an open neighborhood V of z_0 with $R_e(f(z)) = u(z)$ for all z in V.

Let D be the open unit disk in the complex plane with the Kobayashi (=Poincare′) pesudodistance k_D. The Caratheodory pseudodistance c_M of M is defined by

$$c_M(p, q) = \sup\{k_D(f(p), \ f(q)) \text{ for } p, \ q \in M \text{ and } f \in F\},$$

where F denotes the set of holomorphic mappings $f: M \to D$. It is well known that the Kobayashi pseudodistance k_D is eaqual to c_D. For the definition and other Velevant results about pseudodistances refer to [2] or [3].

Let $H = z : z = \{x + ig, \ x > 0\}$ be the right half plane with the Kobayashi pseudodistance k_H. Define

$$p_M(p, q) = sup\{k_H(g(p), \ g(q)) : g \in G\}, \qquad (1)$$

where G denotes the set of all positive harmonic functions g on M.

PROPOSITION 1. *Let D be the unit disk in the complex plane. Then the Kobayashi pseudodistance k_D coincides with the p_D.*

Proof. Let $f: D \to H$ be the biholomorphic map such that $f(0) = 1$ and $f(x) \geq 1$ for $0 \leq x < 1$. Let u be the real part of f. Then it follows from the invariance of k_M for holomorphic maps that

$$p_D(p, q) \geq k_H(u(p), \ u(q)) = k_D(p, q),$$

for any two points in $(-1, 1) \cap D$.

Let A be a holomorphic automorphism of D that maps two points z_1 and

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z_2 of D to two points p and q in the real axis. Then we have the following relations;

$$p_D(z_1, z_2) \geq k_H(u(A(z_1)),\ u(A(z_2))) = k_D(p, q) = k_D(z_1, z_2).$$

In the above, equalities follow from the invariance of the Kobayashi pseudodistance for biholomorphic maps. Let $\alpha : D \to (0, \infty)$ be a positive harmonic function. Then we can define a holomorphic function $F : D \to H$, with it's real part α. We shall show that

$$k_H(\alpha(p),\ \alpha(q)) \leq k_H(F(p),\ F(q)). \tag{2}$$

Let $I(H)$ be the group of biholomorphic transformations of H. Then $I(H)$ contains holomorphic maps, (a) $g(z) = kz$ $(k>0)$, and (b) $h(z) = z + it$ (t=real). Set $F(p) = \alpha(p) + i\beta(p)$ and $F(q) = \alpha(q) + i\beta(q)$. By the invariance of the pseudodistance k_H for the transformations (a) and (b), we have the following equalities:

$$\begin{aligned} k_H(Fp),\ F(q)) &= k_H(\alpha(p), \alpha(q) + i(\beta(q) - \beta(p))) \\ &= k_H(\alpha(p)/\alpha(p), \alpha(q)/\alpha(p) + i(\beta(q) - \beta(p))/\alpha(p)). \end{aligned}$$

Hence in the proof of (2), we may assume that $\alpha(p) = 1$ and $\alpha(p) < \alpha(q)$. We shall prove (1), under the assumption

$$\alpha(p) = 1,\ \beta(p) = 0, \text{ and } \alpha(p) < \alpha(q).$$

Let $A : H \to D$ be the biholomorphic map satisfying $A(1) = 0$, $A(x) = 0$ for $1 < x < \infty$.

Then we have

$$\begin{aligned} &k_D(A(\alpha(p)),\ A(\alpha(q)) = k_H(\alpha(p),\ \alpha(q)), \\ &k_D(A(F(p)),\ A(F(q)) = k_H(F(p)),\ F(q)), \end{aligned}$$

and $A(\alpha(p)) = A(F(p)) = 0$ (since $\alpha(p) = F(p) = 1$).

There is the unique geodesic γ (for the metric k_D) through $A(\alpha(q))$ and $A(F(q))$. The γ is a part of the circle with center c $(c>1)$. Since $A(\alpha(q)) > 0$, it is clear that

$$A(\alpha(q)) \leq |A(F(q))|,$$

and

$$k_D(0, A(\alpha(q)) \leq k_D(0, A(f(q))).$$

From this we have

$$k_H(\alpha(p),\ \alpha(q)) \leq k_H(F(q),\ F(q)).$$

Finally we have

$$k_D(p, q) = k_H(F(p), \ F(q)) \geq k_H(\alpha(p), \ \alpha(q)),$$

and

$$k_D(p, q) \geq p_D(p, q).$$

Let ρ be a non-negative real valued function on $M \times M$. If $\rho(x, y) = \rho(y, x)$, $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ and $\rho(x, x) = 0$ for any three points $\{x, y, z\}$ in S, then ρ is called a *pseudometric* (=*pseudodistance*) on S.

We call any system which assigns a pseudodistance on each complex manifold a *Schwarz-Pick system* if it satisfies the following conditions;

(a) The distance assigned to the unit open disk in the complex plane is the Poincare' metric.

(b) If ρ_1 and ρ_2 are pesudometrics assigned to the complex manifolds S_1 and S_2 respectively, then $\rho_2(h(x), \ h(y)) \leq \rho_1(x, y)$ for all holomorphic mappings $h : S_1 \rightarrow S_2$ and for each pair of points x and y in M.

It is well known that the Kobayashi peudodistance is the largest and the Caratheodory pesudodistance is the smallest one which can be assigned to complex manifolds by a Schwarz-Pick system.

DEFINITION. Let p_M be the real valued function on $M \times M$ defined by (1). We call it PH pseudodistance.

It is clear by proposition 1 that the PH pseudodistance satisfies all of the conditions of the Schwarz-Pick system.

PROPOSITION 2. *The PH pseudodistance p_M is given by a Schwarz-Pick system.*

There is a Riemann surface which carries a positive nonconstant harmonic function but it has no bounded analytic function. Hence the p_M is different from c_M. The p_M is not a metric on compact Riemann surface. But the k_M is known to be a metric on a Riemann surface which is covered by the unit disk. From this we know that p_M and k_M are different.

We call a complex manifold M is p_M *complete* if, for each point p of M and each positive number r, the closed ball of radius $r(\{q : p_M(p, q) \leq r, q \in M\})$ is a compact subset of M. In [3], Kobayashi defines the Caratheodory completenes. He proved that if M is c_M complete then M is F_p convex. Where F_p denotes the set of bounded holomorphic function f on M such that $f(p) = 0$. Let G be the set of positive harmonic functions on M and let $G_p = \{h \in G; \ h(p) = 1\}$. We know from (2) that

$$\begin{aligned} P_M(p, q) &= \sup\{k_H(h(p), \ h(q)) : h \in G\} \\ &= \sup\{c_H(h(p), \ h(q)) : h \in G\} \end{aligned} \tag{3}$$

$$=\sup\{\pm\log h(q) : h\in G_p\}.$$

We define $\hat{h}(z)=\max\{h(z),\ 1/h(z)\}$ for $h\in G_p$. Let K be a subset of M, we set $\hat{k}=\{z\in M : \hat{h}(z)\leq\sup \hat{h}(k)$ for $h\in G_p\}$. Then $\hat{k}$ is a closed subset of M, containing K. If $\hat{k}$ is compact for every compact subset of M, then M is said to be convex with respect to G_p.

PROPOSITION 3. *Let M be a complex manifold. Let G_p be the set of positive harmonic function h on M such that $h(p)=1$ and $p\in M$. If M is complete for p_M then M is convex with respect to G_p.*

Proof. Let r be a positive number. Let $B(r)$ be the ball of radius r around $p\in M$, that is,

$$B(r)=\{q\in M : p_M(p,q)\leq r\}.$$

Since M is complete, $B(r)$ is compact. Let $p\in K$ be a compact subset of M. Then

$$\begin{aligned}\sup\{p_M(p,q) : q\in K\} &\leq\sup\{k_M(p,q) : q\in K\}\\ &=\alpha<\infty.\end{aligned}$$

From the above we know that K is contained in $B(r)$ for sufficiently large r. It suffices to show that $\hat{B}(r)$ is compact. By definition of $\hat{B}(r)$ we have

$$\begin{aligned}\hat{B}(r)&=\{q\in M : \hat{h}(q)\leq\sup_{t\in B(r)}\hat{h}(t),\ \ h\in G_p\}\\ &=\{q\in M : |\log\ h(q)|\leq\sup_{t\in B(r)}|\log\ h(t)|,\ \ h\in G_p\}.\end{aligned}$$

From (3) we know that

$$r=\{\sup_{t\in B(r)}|\log\ h(t)| : h\in G\}$$

and

$$p_M(p,q)=\sup\{|\log\ h(q)| : h\in G_p\}.$$

It follows from the above

$$\hat{B}(r)=\{q\in M : p_M(p,q)\leq r\}.$$

Hence we have $\hat{B}(r)=B(r)$.

In the following we study the PH pseudodistance on Riemann surfaces. Let M be a complex manifold and $p\in M$ then M is said to be Caratheodory complete if for every $r>0$ the closed ball of radius r about p in this distance is compact. Let D be the open unit disk in the complex plane. Let $M_1=\{z : z\in D,\ z\neq 0\}$. Then $-\log|z|$ is a positive harmonic function defined on M_1 and clearly it is not defined on D. Hence we have the following equalities;

$$\lim_{n\to\infty} p_{M_1}\left(\frac{1}{2}, \frac{1}{n}\right) = \infty,$$

and

$$\lim_{n\to\infty} p_D\left(\frac{1}{2}, \frac{1}{n}\right) = p_D\left(\frac{1}{2}, 0\right).$$

Let $M_n = D - \{z_1, z_2, \cdots, z_n\}$. Since the Caratheodory pseudodistance is defined by bounded holomorphic functions, the extension property of bounded holomorphic function implies that $c_{M_m}(p, q) = c_D(p, q)$ for $p, q \in M_m$. From this we know that $M_n (n>0)$ is not complete for the pseudodistance c_{M_n}. On the other hand (7) implies that M is complete for the PH pseudodistance. The same reasoning applies to p_M, and hence M_n is also complete for p_{Mn}. we may state the following proposition.

PROPOSITION 4. *Let D be the unit disk in the complex plane. Let $M_n = D - \{z_1, \cdots, z_n\}$. Then M_n $(n>0)$ is not complete for the PH pseudodistance.*

We shall give another example which shows the difference of c_M and p_M. Let F_p be the set of bounded holomorphic function f on M with $f(p)=0$. Kobayashi show that if M is Caratheodory complete then M is F_p convex. That is, if $K \subseteq M$ is compact then so is

$$\hat{k} = \{q \in M : |f(q)| \leq \sup_{t\in K} |f(t)|,\ f \in F_p\}.$$

Let x_n be a sequence of positive numbers converging monotonically to 0. Let r_n be another sequence of positive numbers such that the closed disks of radius r_n about x_n are pairwise disjoint and such that $\sum r_n (x_n - r_n)^{-1} < \frac{1}{2}$. Let N be the Riemann sphere with 0 and the union of these closed discs removed. In [1], they show that N is F_∞ convex but not complete for the Caratheodory pseudometric. Let D be the open unit disk in the complex plane. Then there is a holomorphic one to one map f which maps N into D. Set $L = f(N)$ and a be in the boundary of L. Then $-\log\left|\frac{1}{2}(z-a)\right|$ is positive and harmonic on L. From this it follows that $L(N)$ is complete for the PH pseudodistance and convex for $G_{f(\infty)} (G_\infty)$

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