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# ON CHARACTERS OF $\eta$-RELATED TENSORS IN COSYMPLECTIC AND SASAKIAN MANIFOLDS (1)

SANG-SEUP EUM*

## 0. Introduction

Recently, S. Tachibana defined in [4] a *conformal Killing tensor* in a Riemannian manifold $M^n$ by a skew symmetric tensor $u_{ji}$ satisfying the equation

$$\nabla_k u_{ji} + \nabla_j u_{ki} = 2\rho_i g_{kj} - \rho_j g_{ki} - \rho_k g_{ji},$$

where $g_{ji}$ is the metric tensor of $M^n$, $\nabla_k$ denotes the covariant derivative with respect to $g_{ji}$ and $\rho_i$ is a covector field. In the present paper, a covector field means a 1-form.

On the other hand, in a Sasakian manifold $M$ with structure tensor $\varphi_j{}^h$, structure vector $\eta^h$ and 1-form $\eta_j$, the following equation is satisfied:

$$\nabla_k \varphi_{ji} + \nabla_j \varphi_{ki} = -(2\eta_i \gamma_{kj} - \eta_j \gamma_{ki} - \eta_k \gamma_{ji}),$$

where $\gamma_{ji} = g_{ji} - \eta_j \eta_i$, $g_{ji}$ being the metric tensor of $M$.

In relation to this fact, we define an *$\eta$-conformal Killing tensor* by a skew symmetric tensor $u_{ji}$ satisfying

$$(*) \qquad \nabla_k u_{ji} + \nabla_j u_{ki} = 2\rho_i \gamma_{kj} - \rho_j \gamma_{ki} - \rho_k \gamma_{ji}$$

in normal almost contact manifolds.

The purpose of the present paper is to analogize the theorems obtained in [4] to normal almost contact manifolds replacing the conformal Killing tensor by the $\eta$-conformal Killing tensor $u_{ji}$ satisfying (*).

In chapter 1, we investigate $\eta$-conformal Killing tensors in a cosymplectic manifold $M'$. In § 1, we state some fundamental formulas in $M'$ to fix our notations and give an example of an $\eta$-conformal Killing tensor in $M'$. In § 2, we discuss the properties of an $\eta$-conformal Killing tensor in $M'$. In § 3, we obtain a condition for a skew symmetric tensor to be an $\eta$-conformal Killing tensor in $M'$. In § 4, we find a unique decomposition of a special $\eta$-conformal Killing tensor in a cosymplectic manifold of constant

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curvature with respect to the tensor $\gamma_{ji}$.

In chapter 2, we investigate $\eta$-conformal Killing tensors in a Sasakian manifold $M$. In §5, we introduce a Sasakian manifold $M$ as a hypersurface in a Kaehlerian manifold $'M$ and fix the second fundamental tensor of $M$ in the case of $M$ is an integral Sasakian hypersurface of a Kaehlerian manifold $'M$ of constant holomorphic sectional curvature which is equal to 4. Moreover in §5, we define an $\eta$-conformal Killing tensor by a skew symmetric tensor $u_{ji}$ satisfying the differential equation (*) and give an example of an $\eta$-conformal Killing tensor. In §6, we discuss the properties of an $\eta$-conformal Killing tensor in $M$. In §7, we obtain a condition for a skew symmetric tensor to be an $\eta$-conformal Killing tensor in $M$. In §8, we find a unique decomposition of a special $\eta$-conformal Killing tensor whose associated vector is a Killing vector in a Sasakian manifold of constant $C$-holomorphic sectional curvature which is equal to 5.

## I. On $\eta$-conformal Killing tensors in a cosymplectic manifold.

### 1. Cosymplectic manifolds of constant curvature with respect to $\gamma_{ji}$.

Let $M$ be a $(2n+1)$-dimensional differentiable manifold of class $C^\infty$ covered by a system of coordinate neighborhoods $\{U;\ x^h\}$ in which there are given a tensor field $\varphi_i{}^h$ of type (1,1), a vector field $\xi^h$ and a 1-form $\eta_i$ satisfying

$$(1.1)\qquad \varphi_j{}^i\varphi_i{}^h=-\delta_j^h+\eta_j\xi^h,\qquad \varphi_j{}^h\xi^j=0,\qquad \eta_i\varphi_j{}^i=0,\qquad \eta_i\xi^i=1$$

and a Riemannian metric $g_{ji}$ satisfying

$$(1.2)\qquad g_{ts}\varphi_j{}^t\varphi_i{}^s=g_{ji}-\eta_j\eta_i,\qquad \eta_i=g_{ih}\xi^h$$

where here and in the sequel the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, 2n+1\}$. Such a set of $\varphi_j{}^h, \xi^h, \eta_j$ and $g_{ji}$ is called an almost contact metric structure and a manifold with an almost contact metric structure an *almost contact metric manifold.*

Comparing the first equation of (1.1) with (1.2), we see that $\varphi_{ji}=\varphi_j{}^t g_{ti}$ is skew symmetric. Since, in an almost contact metric manifold, we have the second equation of (1.2), we shall write $\eta^h$ instead of $\xi^h$ in the sequel.

An almost contact metric manifold $M$ with an almost contact metric structure $(\varphi_j{}^h, \eta^h, \eta_j, g_{ji})$ is called an *almost cosymplectic manifold* if 2-form $\Phi=\varphi_{ji}dx^j\wedge dx^i$ and 1-form $\omega=\eta_j dx^j$ are both closed, that is, $d\Phi=0$ and $d\omega=0$.

If the structure $(\varphi_j{}^h, \eta^h, \eta_j)$ satisfies $N_{ji}{}^h+(\partial_j\eta_i-\partial_i\eta_j)\eta^h=0$, where $N_{ji}{}^h$

is the Nijenhuis tensor formed with $\varphi_i{}^h$, then the structure is said to be *normal.*

A normal almost cosymplectic manifold is said to be a ***cosymplectic manifold.*** It is well known that the cosymplectic structure $(\varphi_j{}^h, \eta^h, \eta_j, g_{ji})$ is characterized by the following two conditions (Blair, [1]):

$$(1.3) \qquad \nabla_j \varphi_{ki}=0, \quad \nabla_j \eta_i=0.$$

By the Ricci identities, the following identities are well known in a cosymplectic manifold:

$$(1.4) \qquad K_{kjt}{}^h \eta^t=0, \quad K_{jt}\eta^t=0,$$

where $K_{kji}{}^h$ and $K_{ji}$ are the curvature tensor and the Ricci tensor of the cosymplectic manifold respectively.

In the present paper, we put

$$(1.5) \qquad \gamma_{ji}=g_{ji}-\eta_j\eta_i.$$

In a previous paper (Yano, Eum and Ki, [8]), we proved that $\gamma_{ji}$ is positive definite for any vector $X^i \neq (\eta_t X^t)\eta^i$.

We define the sectional curvature with respect to $\gamma_{ji}$ of a plane spaned by linearly independent vectors $X^i(\neq (\eta_t X^t)\eta^i)$ and $Y^i$ $(\neq (\eta_t Y^t)\eta^i)$ in $M$ by

$$\rho(X, Y)=-\frac{K_{kjti}X^k Y^j X^t Y^i}{\gamma_{kj}X^k X^j \gamma_{ti} Y^t Y^i-(\gamma_{kj}X^k Y^j)^2}.$$

If $\rho(X, Y)$ is a constant $k$ for any $X^i$ and $Y^i$, then the curvature tensor is of the form

$$(1.6) \qquad K_{kjit}=k(\gamma_{kt}\gamma_{ji}-\gamma_{ki}\gamma_{jt}),$$

where

$$(1.7) \qquad k=\frac{K}{2n(2n-1)},$$

$K$ being the scalar curvature of $M$. In this case, we call $M$ a manifold of ***constant curvature with respect to*** $\gamma_{ji}$.

If $\rho^i$ is a Killing vector in a cosymplectic manifold of constant curvature with respect to $\gamma_{ji}$, then the tensor $q_{ji}$ given by

$q_{ji}=-\frac{1}{c}\nabla_j\rho_i, \quad c=\frac{K}{2n(2n-1)}$ satisfies the following equation:

$$(1.8) \qquad \nabla_k q_{jt}+\nabla_j q_{kt}=2(\rho_t-c'\eta_t)\gamma_{kj}-(\rho_j-c'\eta_j)\gamma_{kt}-(\rho_k-c'\eta_k)\gamma_{jt},$$

where $c'=\rho_t\eta^t$, by virtue of $\nabla_k\nabla_j\rho_i=-K_{tkji}\rho^t$ and (1.6).

Putting $\rho_t - c'\eta_t = '\rho_t$, we see that

$$\nabla_k q_{jt} + \nabla_j q_{kt} = 2'\rho_t \gamma_{kj} - '\rho_j \gamma_{kt} - '\rho_k \gamma_{jt}. \tag{1.9}$$

We define an *$\eta$-conformal Killing tensor* in a cosymplectic manifold by a skew symmetric tensor $q_{ji}$ satisfying the equation (1.9) for a certain covector $'\rho_t$. We call $'\rho_t$ the associated covector of the $\eta$-conformal Killing tensor $q_{ji}$.

A skew symmetric tensor $p_{ji}$ is called a Killing tensor if it satisfies the following equation: $\nabla_k p_{jt} + \nabla_j p_{kt} = 0$.

Thus we have the following

THEOREM 1. *In a cosymplectic manifold of constant curvature with respect to $\gamma_{ji}$, if $K \neq 0$, $\rho_i$ is a Killing covector, $q_{ji} = -\frac{2n}{K}(2n-1)\nabla_j \rho_i$ and $p_{ji}$ is a Killing tensor, then the tensor defined by $p_{ji} + q_{ji}$ is an $\eta$-conformal Killing tensor.*

The converse case of the theorem 1 will be discussed in §4.

## 2. $\eta$-conformal Killing tensors in a cosymplectic manifold.

In this section, we define an $\eta$-conformal Killing tensor by a skew symmetric tensor $u_{ji}$ satisfying the equation

$$\nabla_j u_{it} + \nabla_i u_{jt} = 2\rho_t \gamma_{ji} - \rho_j \gamma_{it} - \rho_i \gamma_{jt} \tag{2.1}$$

in a cosymplectic manifold $M$.

Transvecting (2.1) with $g^{ji}$, we obtain

$$\nabla^r u_{rt} = (2n-1)\rho_t + (\rho_r \eta^r)\eta_t, \tag{2.2}$$

where $\nabla^r$ denotes the operator $g^{rt}\nabla_t$.

By the Ricci identity, we can easily verify that

$$\nabla^r \nabla^t u_{rt} = 0, \tag{2.3}$$

for an arbitrary skew symmetric tensor $u_{ji}$.

In the following, we shall write $\rho_{ji}$ instead of $\nabla_j \rho_i$ for brevity. Operating $\nabla_k$ to (2.1), we obtain

$$\nabla_k \nabla_j u_{it} + \nabla_k \nabla_i u_{jt} = 2\rho_{kt} \gamma_{ji} - \rho_{kj} \gamma_{it} - \rho_{ki} \gamma_{jt}. \tag{2.4}$$

Denoting the left side member of (2.4) by $F_{kjit}$ and forming $F_{kjit} + F_{ikjt} - F_{jikt}$, we obtain

$$\begin{aligned} &2\nabla_k \nabla_i u_{jt} + 2K_{jik}{}^r u_{rt} - K_{ikt}{}^r u_{jr} - K_{kjt}{}^r u_{ir} - K_{ijt}{}^r u_{kr} \\ &= 2(\rho_{kt}\gamma_{ji} + \rho_{it}\gamma_{kj} - \rho_{jt}\gamma_{ik}) + (\rho_{jk} - \rho_{kj})\gamma_{it} + (\rho_{ji} - \rho_{ij})\gamma_{kt} - (\rho_{ki} + \rho_{ik})\gamma_{jt}. \end{aligned} \tag{2.5}$$

Denoting the left side member of (2.5) by $E_{kijt}$, forming $E_{kijt} + E_{kjti} + E_{ktij}$

and taking account of the relation

$$\nabla_k\nabla_i u_{jt}+\nabla_k\nabla_j u_{ti}+\nabla_k\nabla_t u_{ij}=3(\nabla_k\nabla_i u_{jt}+\rho_{kj}\gamma_{it}-\rho_{kt}\gamma_{ji})$$

which follows from (2.1), we obtain

$$(2.6)\quad \begin{aligned} &2\nabla_k\nabla_i u_{jt}+K_{ijk}{}^r u_{tr}+K_{jtk}{}^r u_{ir}+K_{tik}{}^r u_{jr} \\ &=(\rho_{it}-\rho_{ti})\gamma_{kj}+(\rho_{tj}-\rho_{jt})\gamma_{ik}+(\rho_{ji}-\rho_{ij})\gamma_{kt}+2(\rho_{kt}\gamma_{ji}-\rho_{kj}\gamma_{ti}).\end{aligned}$$

By subtracting (2.6) from (2.5), we find

$$(2.7)\quad \begin{aligned} &-K_{jikr}u_t{}^r+K_{tkjr}u_i{}^r-K_{tkir}u_j{}^r+K_{jitr}u_k{}^r \\ &=\sigma_{jk}\gamma_{it}-\sigma_{ki}\gamma_{jt}+\sigma_{it}\gamma_{kj}-\sigma_{tj}\gamma_{ik},\end{aligned}$$

where we have put

$$(2.8)\quad \sigma_{kj}=\rho_{kj}+\rho_{jk}.$$

Operating $\nabla^t$ to (2.2) and taking account of (1.3) and (2.3), we obtain

$$(2.9)\quad (\nabla^t\rho_r)\eta^r\eta_t=-(2n-1)\rho_t{}^t.$$

Transvecting (2.8) with $\gamma^{kj}$ and taking account of (2.9), we obtain

$$(2.10)\quad \gamma^{kj}\sigma_{kj}=2n\sigma_t{}^t=4n\rho_t{}^t.$$

Transvecting (2.7) with $\gamma^{ik}$ and taking account of (1.4) and making use of $(K_{tkjr}+K_{jktr})u^{kr}=0$, we obtain

$$(2.11)\quad -K_{jr}u_t{}^r-K_{tr}u_j{}^r=\sigma_{jk}\gamma_t{}^k+\sigma_{it}\gamma_j{}^i-2n\sigma_{tj}-2n\sigma_r{}^r\gamma_{jt}.$$

Transvecting (2.11) with $g^{jt}$ and taking account of (2.10), we obtain

$$(2.12)\quad \sigma_t{}^t=0,\quad \rho_t{}^t=0.$$

Substituting (2.12) into (2.11) and making use of (1.5), we obtain

$$(2.13)\quad K_{jr}u_t{}^r+K_{tr}u_j{}^r=2(n-1)\sigma_{tj}+\sigma_{jr}\eta_t\eta^r+\sigma_{tr}\eta_j\eta^r.$$

Transvecting (2.13) with $\eta^j$, we obtain

$$(2.14)\quad K_{tr}\eta^s u_s{}^r=(2n-1)\sigma_{tr}\eta^r+(\sigma_{sr}\eta^s\eta^r)\eta_t$$

by virtue of (1.4).

Transvecting (2.14) with $\eta^t$, we obtain

$$(2.15)\quad \sigma_{st}\eta^s\eta^t=0$$

by virtue of (1.4).

Substituting (2.15) into (2.14), we obtain

$$(2.16)\quad \sigma_{tr}\eta^r=\frac{1}{2n-1}K_{tr}\eta^s u_s{}^r.$$

Substituting (2.16) into (2.13), we obtain

$$K_{jr}u_t{}^r+K_{tr}u_j{}^r=2(n-1)\sigma_{tj}+\frac{1}{2n-1}(K_{jr}\eta^s u_s{}^r\eta_t+K_{tr}\eta^s u_s{}^r\eta_j)$$

and from which, in the case of $n>1$,

$$\sigma_{tj}=\frac{1}{2(n-1)}[K_{jr}(\delta_t^s-\frac{1}{2n-1}\eta_t\eta^s)u_s{}^r] +\frac{1}{2(n-1)}[K_{tr}(\delta_j^s-\frac{1}{2n-1}\eta_j\eta^s)u_s{}^r]. \tag{2.17}$$

If we put

$$\alpha_t{}^s=\frac{1}{2(n-1)}(\delta_t^s-\frac{1}{2n-1}\eta_t\eta^s) \tag{2.18}$$

then (2.17) is rewritten as

$$\sigma_{tj}=(K_t{}^r\alpha_j{}^s+K_j{}^r\alpha_t{}^s)u_{sr}. \tag{2.19}$$

Substituting (2.19) into (2.7), we obtain

$$(T_{ijk}{}^r{}_t{}^s-T_{ktj}{}^r{}_i{}^s-T_{tki}{}^r{}_j{}^s+T_{jit}{}^r{}_k{}^s)u_{sr}=0 \tag{2.20}$$

where we have put

$$T_{ijk}{}^r{}_t{}^s=K_{ijk}{}^r\delta_t^s-(K_i{}^r\alpha_t{}^s+K_t{}^r\alpha_i{}^s)\gamma_{jk}. \tag{2.21}$$

In this place, we assume that there exists (locally) an $\eta$-conformal Killing tensor which takes any preassigned (skew symmetric) value at any point of a $(2n+1)$-dimensional cosymplectic manifold $M$ $(n>1)$. Then the skew symmetric parts of the coefficients of $u_{sr}$ in (2.20) vanish. Therefore, we obtain

$$T_{ijk}{}^r{}_t{}^s-T_{ktj}{}^r{}_i{}^s-T_{tki}{}^r{}_j{}^s+T_{jit}{}^r{}_k{}^s -T_{ijk}{}^s{}_t{}^r+T_{ktj}{}^s{}_i{}^r+T_{tki}{}^s{}_j{}^r-T_{jit}{}^s{}_k{}^r=0. \tag{2.22}$$

Contracting with respect to $s$ and $t$ in (2.22) and taking account of

$$K_t{}^r\alpha_i{}^t=\frac{1}{2(n-1)}K_i{}^r, \tag{2.23}$$

which is obtained by virtue of (1.4) and (2.18), we have

$$2nK_{ijk}{}^r-K_{kj}\delta_i^r+K_{ki}\delta_j^r+\left(\frac{1}{2(n-1)}-k\right)(K_i{}^r\gamma_{jk}-K_j{}^r\gamma_{ik}) +(K\gamma_{jk}-K_{jk})\alpha_i{}^r-(K\gamma_{ik}-K_{ik})\alpha_j{}^r=0, \tag{2.24}$$

where we have put $\alpha_t{}^t=k$.

Transvecting (2.24) with $\eta^i\eta_r$ and taking account of (1.4) and (2.18), we obtain

$$K_{kj}=\frac{K}{2n}\gamma_{kj}, \tag{2.25}$$

$K$ being the scalar curvature of $M$.

Substituting (2.25) into (2.24), we obtain

$$\alpha_j{}^r=\frac{1}{2(n-1)(2n-1)}\gamma_j{}^r+\frac{1}{2n-1}\delta_j{}^r. \tag{2.26}$$

Substituting (2.25) and (2.26) into (2.24), we obtain

$$K_{kjit}=\frac{K}{2n(2n-1)}(\gamma_{kt}\gamma_{ji}-\gamma_{ki}\gamma_{jt}) \tag{2.27}$$

Thus we have the following

THEOREM 2. *If there exists (locally) an $\eta$-conformal Killing tensor which takes any preassigned (skew symmetric) value at any point of a $(2n+1)$-dimensional cosymplectic manifold $M$ $(n>1)$, then $M$ is a manifold of constant curvature with respect to $\gamma_{ji}$.*

Transvecting (2.27) with $g^{kt}$, we obtain (2.25), that is,

$$K_{ji}=\frac{K}{2n}\gamma_{ji}. \tag{2.28}$$

On the other hand, the conformal curvature tensor of Weyl in a $(2n+1)$-dimensional Riemannian manifold is defined for $n>1$ by

$$C_{kji}{}^h=K_{kji}{}^h+\frac{1}{2n-1}(L_{ki}\delta_j{}^h-L_{ji}\delta_k{}^h+g_{ki}L_j{}^h-g_{ji}L_k{}^h), \tag{2.29}$$

where $L_{ji}=K_{ji}-\frac{K}{4n}g_{ji}$.

Substituting (2.27) and (2.28) into (2.29), we easily obtain $C_{kji}{}^h=0$. Thus we have the following

COROLLARY. *If there exists (locally) an $\eta$-conformal Killing tensor which takes any preassigned (skew symmetric) value at any point of a $(2n+1)$-dimensional cosymplectic manifold $M$ $(n>1)$, then $M$ is conformally flat.*

## 3. A condition to be an $\eta$-conformal Killing tensor.

Let $u_{ji}$ be an $\eta$-conformal Killing tensor in a cosymplectic manifold $M$. Then we obtain

$$\eta^j\eta^i\nabla_j u_{it}=0 \tag{3.1}$$

by virtue of (1.5) and (2.1).

If we transvect (2.5) with $g^{ki}$, then we obtain

$$(3.2)\qquad \begin{aligned} &\nabla^r\nabla_r u_{jt}+K_j{}^r u_{rt}+K_{kjt}{}^r u_r{}^k \\ &=-(2n-3)\rho_{jt}-\rho_{tj}-\eta^k(2\eta_j\rho_{kt}-\eta_t\rho_{kj}+\eta_t\rho_{jk}). \end{aligned}$$

In this section, we shall show that a skew symmetric tensor $u_{ji}$ satisfying (3.1) and (3.2) is an $\eta$-conformal Killing tensor in a cosymplectic manifold $M$ provided that $M$ is compact.

For this purpose, we define a tensor $A_{jit}$ by

$$(3.3)\qquad A_{jit}=\nabla_j u_{it}+\nabla_i u_{jt}-2\rho_t\gamma_{ji}+\rho_j\gamma_{it}+\rho_i\gamma_{jt}$$

for a skew symmetric tensor $u_{ji}$, where $\rho_t$ is given by

$$(2n-1)\rho_t=\nabla^r u_{rt}-\frac{1}{2n}(\eta^k\nabla^r u_{rk})\eta_t.$$

By a direct computation, we obtain from (3.3) that

$$(3.4)\qquad \begin{aligned} \nabla^j A_{jit}=&\nabla^j\nabla_j u_{it}+(2n-3)\rho_{it}+\rho_{ti}+\eta_t(\rho_{ik}-\rho_{ki})\eta^k \\ &-2\eta_i\rho_{kt}\eta^k-K^j{}_{it}{}^s u_{js}-K_i{}^s u_{ts}, \end{aligned}$$

where we have put $\rho_{it}=\nabla_i\rho_t$.

On the other hand, simple computations give us the following equation

$$(3.5)\qquad A_{jit}A^{jit}=2A_{jit}\nabla^j u^{it}+6\rho^t\eta^j\eta^i\nabla_j u_{it}.$$

Substituting (3.4) and (3.5) into $\nabla^j(A_{jit}u^{it})=A_{jit}\nabla^j u^{it}+(\nabla^j A_{jit})u^{it}$, we can obtain

$$(3.6)\qquad \begin{aligned} \nabla^j(A_{jit}u^{it})=&\frac{1}{2}A_{jit}A^{jit}-3\rho^t\eta^j\eta^i\nabla_j u_{it}+u^{it}[\nabla^j\nabla_j u_{it}+(2n-3)\rho_{it} \\ &+\rho_{ti}+\eta_t(\rho_{ik}-\rho_{ki})\eta^k+2\eta_i\rho_{kt}\eta^k+K_{kit}{}^s u_s{}^k+K_i{}^s u_{st}]. \end{aligned}$$

Thus we have the following

THEOREM 3. *In a compact cosymplectic manifold M, the following integral formula is valid for any skew symmetric tensor field* $u_{ji}$:

$$(3.7)\qquad \begin{aligned} \int_M[&u^{it}\{\nabla^j\nabla_j u_{it}+(2n-3)\rho_{it}+\rho_{ti}+\eta_t(\rho_{ik}-\rho_{ki})\eta^k+2\eta_i\rho_{kt}\eta^k \\ &+K_{kit}{}^s u_s{}^k+K_i{}^s u_{st}\}+\frac{1}{2}A_{jit}A^{jit}-3\rho^t\eta^j\eta^i\nabla_j u_{it}]dV=0. \end{aligned}$$

*where dV means the volume element of M and*

$$(2n-1)\rho_{it}=\nabla_i\nabla^r u_{rt}-\frac{1}{2n}(\eta^k\nabla_i\nabla^r u_{rk})\eta_t.$$

Taking account of (3.1), (3.2) and (3.7), we obtain the following

THEOREM 4. *In a compact cosymplectic manifold M, a necessary and sufficient condition for a skew symmetric tensir* $u_{ji}$ *to be an* $\eta$*-conformal Killing tensor is that* (3. 1) *and* (3. 2).

## 4. $\eta$-conformal Killing tensor in a cosymplectic manifold of constant curvature with respect to $\gamma_{ji}$.

In this section, we consider an $\eta$-conformal Killing tensor $u_{ji}$ in a cosymplectic manifold $M$ of constant curvature with respect to $\gamma_{ji}$. In this case, we obtain from (1. 6) that

$$K_{ji}=(2n-1)k\gamma_{ji}, \tag{4.1}$$

$k$ being a constant.

Substituting (4. 1) into (2. 16), we obtain

$$\eta^r\sigma_{rj}=k\eta^r u_{rj}. \tag{4.2}$$

Substituting (4. 1) into (2. 17), we obtain

$$\sigma_{rj}=k(\eta_r\eta^t u_{tj}+\eta_j\eta^t u_{tr}). \tag{4.3}$$

Taking account of (2. 8), (4. 2) and (4. 3), we see that the associated vector $\rho^k$ of $u_{ji}$ is a Killing vector if and only if

$$\eta^r u_{rj}=0. \tag{4.4}$$

In this place, we consider only an $\eta$-conromal Killing tensor $u_{ji}$ whose associated vector $\rho^k$ is a Killing vector.

Under this consideration, we obtain (4. 4) and from which

$$\eta_t\rho^t=0 \tag{4.5}$$

by virtue of (2. 2). Moreover we obtain

$$\nabla_k\nabla_j\rho_i+K_{tkji}\rho^t=0 \tag{4.6}$$

since $\rho^k$ is a Killing vector.

Substituting (1. 6) and (1. 7) into (4. 6) and taking account of (4. 5), we see that

$$\nabla_k\nabla_j\rho_t+\nabla_j\nabla_k\rho_t=-c(2\rho_t\gamma_{kj}-\rho_j\gamma_{kt}-\rho_k\gamma_{jt}), \tag{4.7}$$

where $c=\dfrac{K}{2n(2n-1)}$.

Thus $\nabla_k\rho_j$ is an $\eta$-conformal Killing tensor.

In this case, if we put

(4.8) $$p_{ji}=u_{ji}+\frac{1}{c}\nabla_j\rho_i,$$

then $\eta^j p_{ji}=0$ by virtue of (4.4) and (4.5). Moreover we obtain $\nabla_k p_{ji}+\nabla_j p_{ki}=0$ by virtue of (2.1) and (4.7). Putting

(4.9) $$q_{ji}=-\frac{1}{c}\nabla_j\rho_i,$$

we see that an $\eta$-conformal Killing tensor $u_{ji}$ is decomposed in the form

(4.10) $$u_{ji}=p_{ji}+q_{ji},$$

where $p_{ji}$ is a Killing tensor such that $\eta^j p_{ji}=0$ and $q_{ji}$ is a closed $\eta$-conformal Killing tensor by virtue of (4.6).

The uniqueness of the decomposition (4.10) is proved by the following way. In fact, if

$$u_{ji}=p_{ji}+q_{ji}={}'p_{ji}+{}'q_{ji}$$

where $p_{ji}$ and $'p_{ji}$ are Killing tensors and $q_{ji}$ and $'q_{ji}$ are closed $\eta$-conformal Killing tensors, then putting

$$P_{ji}={}'p_{ji}-p_{ji}=q_{ji}-{}'q_{ji},$$

we see that $P_{ji}$ is a closed Killing tensor. Therefore we have

$$\nabla_k P_{ji}+\nabla_i P_{kj}+\nabla_j P_{ik}=0,\quad \nabla_k P_{ji}+\nabla_j P_{ki}=0,$$

and from which we can obtain $\nabla_k P_{ji}=0$. Thus by virtue of Ricci identity it follows that

(4.11) $$K_{tkj}{}^s P_{si}+K_{tki}{}^s P_{js}=0.$$

Since $p_{ji}\eta^i=0$ and $'p_{ji}\eta^i=0$, we obtain

(4.12) $$P_{ji}\eta^i=0.$$

Substituting (1.6), (1.7) and (4.12) into (4.11), we see that

$$\gamma_{kj}P_{ti}-\gamma_{tj}P_{ki}+P_{jt}\gamma_{ki}-P_{jk}\gamma_{ti}=0.$$

Transvecting this equation with $g^{ti}$ and taking account of (4.12) and the fact that $P_{ti}$ is skew symmetric, we obtain $P_{ji}=0$, that is,

$$'p_{ji}-p_{ji}=0,\quad q_{ji}-{}'q_{ji}=0.$$

Thus we have the following

THEOREM 5. *In a cosymplectic manifold of constant curvature with respect to $\gamma_{kj}$, an $\eta$-conformal Killing tensor $u_{ji}$ whose associated vector $\rho^k$ is a Killing vector is uniquely decomposed in the form*

$$u_{ji}=p_{ji}+q_{ji},$$

*where* $p_{ji}$ *is a Killing tensor such that* $\eta^j p_{ji}=0$ *and* $q_{ji}$ *is a closed* $\eta$*-conformal Killing tensor.*

*In this case,* $q_{ji}$ *is the form*

$$q_{ji}=-\frac{1}{c}\nabla_j\rho_i, \quad c=\frac{K}{2n(2n-1)}.$$

## II. On $\eta$-conformal Killing tensors in a Sasakian manifold.

### 5. A Sasakian manifold as a hypersurface in a Kaehlerian manifold.

We consider a $(2n+2)$-dimensional Kaehlerian manifold $'M$ covered by a system of coordinate neighborhoods $\{'U; y^\lambda\}$, where here and in the sequel the indices $\lambda, \mu, \nu, \tau, \ldots$ run over the range $\{1, 2, \ldots, 2n+2\}$. We denote by $G_{\mu\lambda}$ and $F_\mu{}^\lambda$ the components of the Kaehlerian metric tensor and those of the Kaehlerian structure tensor of $'M$ respectively. In this place, let $M$ be a real hypersurface in a Kaehlerian manifold $'M$. If we assume that $M$ is covered by a system of coordinate neighborhoods $\{U; x^h\}$, where $U='U\cap M$, then $M$ is represented by $y^\lambda=y^\lambda(x^h)$ in terms of local coordinates $(y^\lambda)$ in $'U(\subset 'M)$ and $(x^h)$ in $U(\subset M)$. We denote the vectors $\partial_h y^\lambda$ $(\partial_h=\partial/\partial x^h)$ tangent to $M$ by $B_h{}^\lambda$ and the unit normal vector by $N^\lambda$. Then by the relations

$$(5.1) \qquad F_\mu{}^\lambda B_h{}^\mu=\varphi_h{}^i B_i{}^\lambda+\eta_h N^\lambda, \qquad F_\mu{}^\lambda N^\mu=-\eta^h B_h{}^\lambda,$$

it is well known that the set $(\varphi_j{}^h, \eta^h, \eta_j, g_{ji})$ becomes an almost contact metric structure. If this almost contact metric manifold satisfies

$$(5.2) \qquad \varphi_{ji}=\frac{1}{2}(\partial_j\eta_i-\partial_i\eta_j), \quad (\varphi_{ji}=\varphi_j{}^t g_{ti})$$

then this structure is called a *contact structure.* Moreover, a manifold with a normal contact structure is called a *Sasakian manifold.*

It is well known that in above Sasakian manifold $M$ the following equations are satisfied:

$$(5.3) \qquad \nabla_j\eta_i=\varphi_{ji},$$

$$(5.4) \qquad \nabla_h\varphi_{ji}=-g_{hj}\eta_i+g_{hi}\eta_j,$$

where $\nabla_j$ denotes the operators of covariant differentiation with respect to $g_{ji}$. (Yano, [7])

We consider an almsot contact metric manifold $M$ as a hypersurface in a Kaehlerian manifold $'M$. Then the equations of Gauss and Weingarten are respectively

$$\nabla_j B_i^\lambda = h_{ji} N^\lambda, \quad \nabla_j N^\lambda = -h_j^t B_t^\lambda, \tag{5.5}$$

where $h_{ji}$ is the second fundamental tensor of $M$ with respect to the normal vector $N^\lambda$ and $h_j^t = h_{ji} g^{it}$.

Differentiating covariantly (5.1) and taking account of $\nabla_i = B_i^\lambda \nabla_\lambda$, we see that

$$\nabla_k \varphi_{ji} = -h_{kj}\eta_i + h_{ki}\eta_j, \tag{5.6}$$

$$\nabla_j \eta_i = -\varphi_i^t h_{tj}. \tag{5.7}$$

Taking account of (5.3), (5.4), (5.6) and (5.7), we easily verify that the following

PROPOSITION 6. *In order that the hypersurface $M$ in a Kaehlerian manifold $'M$ be a Sasakian manifold, it is necessary and sufficient that the second fundamental tensor $h_{ji}$ of $M$ has the form*

$$h_{ji} = g_{ji} + \mu \eta_j \eta_i, \tag{5.8}$$

*where $\mu$ is a scalar field in $M$.* (Tashiro and Tachibana [6])

If the Kaehlerian manifold $'M$ has constant holomorphic sectional curvature $c$ at each point of $'M$, then the curvature tensor $'K_{\tau\nu\mu\lambda}$ of $'M$ has the form

$$'K_{\tau\nu\mu\lambda} = \frac{c}{4}(G_{\tau\lambda}G_{\nu\mu} - G_{\nu\lambda}G_{\tau\mu} + F_{\tau\lambda}F_{\nu\mu} - F_{\nu\lambda}F_{\tau\mu} - 2F_{\tau\nu}F_{\mu\lambda}). \tag{5.9}$$

Such a Kaehlerian manifold $'M$ is called locally *Fubinian* (Tashiro and Tachibana [6]). A locally Fubinian manifold is an Einstein space, that is, the Ricci tensor $'K_{\mu\lambda}$ of $'M$ has the form:

$$'K_{\mu\lambda} = \frac{1}{2}(n+2)cG_{\mu\lambda}. \tag{5.10}$$

Substituting (5.8) and (5.9) into the equation of Codazzi:

$$\nabla_k h_{ji} - \nabla_j h_{ki} = {'K}_{\nu\mu\lambda\tau} B_k^\nu B_j^\mu B_i^\lambda N^\tau,$$

we see that $\mu = -\frac{c}{4}$ by vritue of (5.9) and (5.10).

Therefore we have the following

PROPOSITION 7. *Let $M$ be a hypersurface in a Kaehlerian manifold of constant holomorphic sectional curvature $c$. If $M$ is a Sasakian manifold, then the second fundamental tensor of $M$ has the form:*

(5. 11) $$h_{ji}=g_{ji}-\frac{c}{4}\eta_j\eta_i.$$

Substituting (5. 9) and (5. 11) into the equation of Gauss:

$$K_{kjih}='K_{\nu\mu\lambda\tau}B_k{}^\nu B_j{}^\mu B_i{}^\lambda B_h{}^\tau+(h_{kh}h_{ji}-h_{jh}h_{ki}),$$

where $K_{kjih}$ is the curvature tensor of $M$, we obtain

(5. 12) $$K_{kjih}=\left(\frac{c}{4}+1\right)(g_{kh}g_{ji}-g_{jh}g_{ki})+\frac{c}{4}(\varphi_{kh}\varphi_{ji}-\varphi_{jh}\varphi_{ki}-2\varphi_{kj}\varphi_{ih})$$
$$-\frac{c}{4}(g_{ji}\eta_k\eta_h+g_{kh}\eta_j\eta_i-g_{jh}\eta_k\eta_i-g_{ki}\eta_j\eta_h),$$

that is, the $C$-holomorphic sectional curvature of $M$ is constant which is equal to $c+1$. (Ogiue, [2]) Such a Sasakian manifold is called a locally *C-Fubinian manifold.* (Tashiro and Tachibana ([6])

We consider the system of partial differential equation in a Kaehlerian manifold $'M$,

(5. 13) $$\nabla_\mu X_\lambda=\frac{c}{4}(G_{\mu\lambda}+X_\mu X_\lambda-F_\mu{}^\tau X_\tau F_\lambda{}^\nu X_\nu),$$

for an arbitrary vector $X^\lambda$ in $'M$.

It is well known that the necessary and sufficient condition for the system (5. 13) to be completely integrable is that the curvature tensor of $'M$ has the form (5. 9). Therefore if $'M$ is a locally Fubinian and the curvature tensor has the form (5. 9), then there exists locally in $'M$ a vector field $X^\lambda$ satisfying (5. 13). Let $P$ be an arbitrary point in $'M$ and consider a solution of (5. 13) with arbitrary initial value $(X_\lambda)_P$ at $P$ satisfying $(G_{\mu\lambda}X^\mu X^\lambda)_P=1$. Putting $\omega=X_\lambda dy^\lambda$, we find $d\omega=0$ because of $\nabla_\mu X_\lambda=\nabla_\lambda X_\mu$. Thus the Pfaffian equation $\omega=0$ is completely integrable. Let $M$ be the integral manifold of $\omega=0$ passing through $P$. Since $X^\lambda$ is a normal vector, we can put

(5. 14) $$X^\lambda=-\alpha N^\lambda,$$

where $\alpha$ is a function in $M$.

We consider this integral manifold $M$ as a hypersurface satisfying (5. 11) in $'M$.

Substituting (5. 14) into $B_j{}^\mu B_i{}^\lambda\nabla_\mu X_\lambda$ and taking account of the second equation of (5. 5), we obtain

(5. 15) $$B_j{}^\mu B_i{}^\lambda\nabla_\mu X_\lambda=\alpha h_{ji}.$$

On the other hand, substituting (5. 13) into $B_j{}^\mu B_i{}^\lambda\nabla_\mu X_\lambda$ and taking account of the first equation of (5. 1), we obtain

$$B_j{}^{\mu}B_i{}^{\lambda}\nabla_{\mu}X_{\lambda}=\frac{c}{4}(g_{ji}-\alpha^2\eta_j\eta_i). \tag{5.16}$$

Comparing (5.15) with (5.16), we see that

$$h_{ji}=\nu g_{ji}-\alpha\eta_j\eta_i, \tag{5.17}$$

where we have put $\nu=\frac{c}{4\alpha}$.

In order that the condition (5.11) is compatible with the condition (5.17), we should have $\nu=1$ and $\alpha=\frac{c}{4}$. On the other hand, by means of the initial condition we find $\alpha^2=1$. Thus we may suppose that $\alpha=1$. In this case $c=4$ and the condition (5.17) is rewritten as

$$h_{ji}=\gamma_{ji}, \tag{5.18}$$

where we have put $\gamma_{ji}=g_{ji}-\eta_j\eta_i$.
Taking account of (5.12) and (5.18), we have the following

THEOREM 8. *In a Kaehlerian manifold $'M$ of constant holomorphic sectional curvature, there exists locally an integral hypersurface $M$ of $X_\lambda dy^\lambda=0$ such that $M$ is a C–Fubinian manifold, the second fundamental tensor $h_{ji}$ of $M$ has the form $h_{ji}=\gamma_{ji}$ and the curvature tensor $K_{kjih}$ of $M$ has the form*

$$\begin{aligned}K_{kjih}=&\gamma_{kh}\gamma_{ji}-\gamma_{jh}\gamma_{ki}+\varphi_{kh}\varphi_{ji}-\varphi_{jh}\varphi_{ki}-2\varphi_{kj}\varphi_{ih}\\&+(g_{kh}g_{ji}-g_{jh}g_{ki}),\end{aligned} \tag{5.19}$$

*that is, the C–holomorphic sectional curvature is equal to* 5.

Next, we consider a parallel skew symmetric tensor field $v_{\mu\lambda}$ in a Kaehlerian manifold $'M$ of constant holomorphic sectional curvature. In fact $F_{\mu\lambda}$ is such a tensor field. We define a tensor field $u_{ji}$ in $M$ satisfying (5.18) by $u_{ji}=B_j{}^{\mu}B_i{}^{\lambda}v_{\mu\lambda}$. Operating $\nabla_k$ to this equation, we have

$$\nabla_k u_{ji}=v_{\mu\lambda}(h_{kj}N^{\mu}B_i{}^{\lambda}+h_{ki}N^{\lambda}B_j{}^{\mu}).$$

If we put $\rho_i=v_{\mu\lambda}N^{\mu}B_i{}^{\lambda}$, then it follows that

$$\nabla_k u_{ji}=h_{kj}\rho_i-h_{ki}\rho_j.$$

Substituting (5.18) in to this equation, we obtain

$$\nabla_k u_{ji}+\nabla_j u_{ki}=2\gamma_{kj}\rho_i-\gamma_{ki}\rho_j-\gamma_{ji}\rho_k. \tag{5.20}$$

If we take $F_{\mu\lambda}$ instead of $v_{\mu\lambda}$, then by means of $B_j{}^{\mu}B_i{}^{\lambda}F_{\mu\lambda}=\varphi_{ji}$ we see that $F_{\mu\lambda}N^{\mu}B_i{}^{\lambda}=-\eta_i$. Thus we have

$$\nabla_k\varphi_{ji}+\nabla_j\varphi_{ki}=-2\gamma_{kj}\eta_i+\gamma_{ki}\eta_j+\gamma_{ji}\eta_k. \tag{5.21}$$

In relation to this fact, we prepare the following

DEFINITION. We call a skew symmetric tensor $u_{ji}$ ($u_{ji} \neq c\varphi_{ji}$) in a Sasakian manifold $M$ an *$\eta$-conformal Killing tensor* if there exists a vector field $\rho^i$ satisfying (5.20). In this case, we call $\rho^i$ the *associated vector* of $u_{ji}$.

## 6. $\eta$-conformal Killing tensors in a Sasakian manifold.

In the last section, we defined an $\eta$-conformal Killing tensor in a Sasakian manifold $M$ (dim $M=2n+1$) by a skew symmetric tensor $u_{ji}$ satisfying the equation

$$\nabla_j u_{it} + \nabla_i u_{jt} = 2\rho_t \gamma_{ji} - \rho_j \gamma_{it} - \rho_i \gamma_{jt}. \tag{6.1}$$

Transvecting (6.1) with $g^{ji}$, we obtain

$$\nabla^r u_{rt} = (2n-1)\rho_t + (\rho_r \eta^r)\eta_t, \tag{6.2}$$

where $\nabla^r$ denotes the operator $g^{rt}\nabla_t$.

By the Ricci identity, we can easily verify that

$$\nabla^r \nabla^t u_{rt} = 0 \tag{6.3}$$

for an arbitrary skew symmetric tensor $u_{ji}$.

In the following we shall write $\rho_{ji}$ instrad of $\nabla_j \rho_i$ for brevity.

Operating $\nabla_k$ to (6.1), we obtain

$$\begin{aligned} \nabla_k \nabla_j u_{it} + \nabla_k \nabla_i u_{jt} = {} & 2\rho_{kt}\gamma_{ji} - \rho_{kj}\gamma_{it} - \rho_{ki}\gamma_{jt} + \varphi_{kt}(\rho_j \eta_i + \rho_i \eta_j) \\ & - \varphi_{kj}(2\rho_t \eta_i - \rho_i \eta_t) - \varphi_{ki}(2\rho_t \eta_j - \rho_j \eta_t). \end{aligned} \tag{6.4}$$

Denoting the left side member of (6.4) by $G_{kjit}$ and forming $G_{kjit} + G_{ikjt} - G_{jikt}$, we obtain

$$\begin{aligned} & 2\nabla_k \nabla_i u_{jt} + 2K_{jik}{}^r u_{rt} - K_{ikt}{}^r u_{jr} - K_{kjt}{}^r u_{ir} - K_{ijt}{}^r u_{kr} \\ & = 2(\rho_{kt}\gamma_{ji} + \rho_{it}\gamma_{kj} - \rho_{jt}\gamma_{ik}) + (\rho_{jk} - \rho_{kj})\gamma_{it} + (\rho_{ji} - \rho_{ij})\gamma_{kt} \\ & \quad - (\rho_{ki} + \rho_{ik})\gamma_{jt} + \varphi_{kt}(\rho_j \eta_i + \rho_i \eta_j) + \varphi_{it}(\rho_k \eta_j + \rho_j \eta_k) \\ & \quad - \varphi_{jt}(\rho_i \eta_k + \rho_k \eta_i) - 2\varphi_{kj}(2\rho_t \eta_i - \rho_i \eta_t) - 2\varphi_{ij}(2\rho_t \eta_k - \rho_k \eta_t). \end{aligned} \tag{6.5}$$

Denoting the left side member of (6.5) by $H_{kijt}$, forming $H_{kijt} + H_{kjti} + H_{ktij}$ and taking account of the relation

$$\begin{aligned} \nabla_k(\nabla_i u_{jt} + \nabla_j u_{ti} + \nabla_t u_{ij}) = 3[ & \nabla_k \nabla_i u_{jt} + (\rho_{kj}\gamma_{it} - \rho_{kt}\gamma_{ji}) \\ & + (\varphi_{kj}\rho_t - \varphi_{kt}\rho_j)\eta_i + \varphi_{ki}(\rho_t \eta_j - \rho_j \eta_t)] \end{aligned}$$

which follows from (6.1), we obtain

$$\begin{aligned} & 2\nabla_k \nabla_i u_{jt} + K_{ijk}{}^r u_{tr} + K_{jtk}{}^r u_{ir} + K_{tik}{}^r u_{jr} \\ & = (\rho_{it} - \rho_{ti})\gamma_{kj} + (\rho_{tj} - \rho_{jt})\gamma_{ik} + (\rho_{ji} - \rho_{ij})\gamma_{kt} + 2(\rho_{kt}\gamma_{ji} - \rho_{kj}\gamma_{ti}) \\ & \quad - 3(\varphi_{kj}\rho_t - \varphi_{kt}\rho_j)\eta_i - \varphi_{ki}(\rho_t \eta_j - \rho_j \eta_t) + (\varphi_{kj}\eta_t - \varphi_{kt}\eta_j)\rho_i \\ & \quad + 2(\varphi_{it}\rho_j + \varphi_{tj}\rho_i + \varphi_{ji}\rho_t)\eta_k. \end{aligned} \tag{6.6}$$

By subtraction (6.5) from (6.6), we find

$$(6.7)\qquad K_{jik}{}^{r}u_{tr}-K_{kti}{}^{r}u_{jr}-K_{tkj}{}^{r}u_{ir}-K_{jit}{}^{r}u_{kr}$$
$$=\sigma_{tj}\gamma_{ik}-\sigma_{it}\gamma_{kj}+\sigma_{ki}\gamma_{jt}-\sigma_{kj}\gamma_{ti}+\varphi_{kj}\alpha_{ti}-\varphi_{ki}\alpha_{tj}+\varphi_{it}\alpha_{jk}$$
$$-\varphi_{jt}\alpha_{ik}+2(\varphi_{kt}\alpha_{ji}-\varphi_{ji}\alpha_{kt}),$$

where we have put

$$(6.8)\qquad \sigma_{tj}=\rho_{tj}+\rho_{jt},\qquad \alpha_{tj}=\rho_{t}\eta_{j}-\rho_{j}\eta_{t}.$$

Operating $\nabla^{t}$ to (6.2) and taking account of (5.3) and (6.3), we obtain

$$(6.9)\qquad \rho_{tr}\eta^{r}\eta^{t}=-(2n-1)\rho_{t}{}^{t}.$$

Transvecting the first equation of (6.8) with $\gamma^{jk}$ and taking account of (6.9), we obtain

$$(6.10)\qquad \gamma^{jk}\sigma_{jk}=4n\rho_{t}{}^{t}=2n\sigma_{t}{}^{t},$$

and from which

$$(6.11)\qquad \sigma_{jk}\eta^{j}\eta^{k}=-(2n-1)\sigma_{t}{}^{t}.$$

On the other hand, transvecting (6.7) with $\gamma^{ik}$ and taking account of the fact that $K_{kjht}\eta^{t}=\gamma_{jh}\eta_{k}-\gamma_{kh}\eta_{j}$, which follows from the Ricci identity $\nabla_{k}\nabla_{j}\eta_{h}-\nabla_{j}\nabla_{k}\eta_{h}=-K_{kjh}{}^{t}\eta_{t}$, we obtain

$$(6.12)\qquad K_{j}{}^{r}u_{tr}+K_{t}{}^{r}u_{jr}=2n\sigma_{tj}-\sigma_{rt}\gamma_{j}{}^{r}+2n\sigma_{r}{}^{r}\gamma_{jt}-\sigma_{rj}\gamma_{t}{}^{r}-3(\varphi_{j}{}^{r}\alpha_{tr}+\varphi_{t}{}^{r}\alpha_{jr}).$$

Transvecting (6.12) with $g^{jt}$, we obtain

$$(6.13)\qquad (2n^{2}-n-1)\sigma_{r}{}^{r}+\sigma_{jk}\eta^{j}\eta^{k}=0.$$

Comparing (6.11) with (6.13), we obtain

$$(6.14)\qquad \sigma_{r}{}^{r}=0,\qquad \sigma_{rt}\eta^{r}\eta^{t}=0.$$

Taking account of (6.8) and (6.14), we also obtain

$$(6.15)\qquad \rho_{t}{}^{t}=0,\qquad \rho_{rt}\eta^{r}\eta^{t}=0.$$

Substituting (6.8) and (6.14) into (6.12), we obtain

$$(6.16)\qquad K_{j}{}^{r}u_{tr}+K_{t}{}^{r}u_{jr}=2(n-1)\sigma_{tj}+(\sigma_{rt}\eta_{j}+\sigma_{rj}\eta_{t})\eta^{r}+3(\varphi_{j}{}^{r}\eta_{t}+\varphi_{t}{}^{r}\eta_{j})\rho_{r}.$$

We shall use these equations in §8.

## 7. A condition to be an $\eta$-conformal Killing tensor.

Let $u_{ji}$ be an $\eta$-conformal Killing tensor in a Sasakian manifold $M$, then we can obtain

$$(7.1)\qquad \eta^{j}\eta^{i}\nabla_{j}u_{it}=0$$

by virtue of (6. 1).

If we transvect (6. 5) with $g^{ki}$ then we obtain

$$(7.2)\qquad \begin{aligned}&\nabla^r\nabla_r u_{jt}+K_j{}^r u_{rt}+K_{kjt}{}^r u_r{}^k\\&=-(2n-3)\rho_{jt}-\rho_{tj}-\eta^k(2\rho_{kt}\eta_j-\eta_t\rho_{kj}+\eta_t\rho_{jk})\\&+\rho^k(2\varphi_{kj}\eta_t-\varphi_{jt}\eta_k+\varphi_{kt}\eta_j).\end{aligned}$$

In this section we shall show that a skew symmetric tensor $u_{ji}$ satisfying (7. 1) and (7. 2) is an $\eta$-conformal Killing tensor in a Sasakian manifold $M$ provided that $M$ is compact.

For this purpose, we define a tensor $A_{jit}$ by

$$(7.3)\qquad A_{jit}=\nabla_j u_{it}+\nabla_i u_{jt}-2\rho_t\gamma_{ji}+\rho_j\gamma_{it}+\rho_i\gamma_{jt},$$

for a skew symmetric tensor $u_{ji}$, where $\rho_t$ is given by

$$(2n-1)\rho_t=\nabla^r u_{rt}-\frac{1}{2n}(\eta^k\nabla^r u_{rk})\eta_t.$$

By a direct computation, we obtain from (7. 3) that

$$(7.4)\qquad \begin{aligned}\nabla^j A_{jit}=&\nabla^j\nabla_j u_{it}+(2n-3)\rho_{it}+\rho_{ti}+\eta^k(2\rho_{kt}\eta_i-\eta_t\rho_{ki}+\eta_t\rho_{ik})\\&-\rho^k(2\varphi_{ki}\eta_t-\varphi_{it}\eta_k+\varphi_{kt}\eta_i)+K_{kit}{}^r u_r{}^k+K_i{}^r u_{rt},\end{aligned}$$

where we have put $\nabla_i\rho_t=\rho_{it}$.

On the other hand, simple computations give us the following equation

$$(7.5)\qquad A_{jit}A^{jit}=2A_{jit}\nabla^j u^{it}+6\rho^t\eta^j\eta^i\nabla_j u_{it}.$$

Substituting (7. 4) and (7. 5) into

$$\nabla^j(A_{jit}u^{it})=A_{jit}\nabla^j u^{it}+(\nabla^j A_{jit})u^{it},$$

we can obtain

$$(7.6)\qquad \begin{aligned}\nabla^j(A_{jit}u^{it})=&\frac{1}{2}A_{jit}A^{jit}-3\rho^t\eta^j\eta^i\nabla_j u_{it}\\&+u^{it}[\nabla^j\nabla_j u_{it}+(2n-3)\rho_{it}+\eta^k(2\rho_{kt}\eta_i-\eta_t\rho_{ki}+\eta_t\rho_{ik})\\&-\rho^k(2\varphi_{ki}\eta_t-\varphi_{it}\eta_k+\varphi_{kt}\eta_i)+K_{kit}{}^r u_r{}^k+K_i{}^r u_{rt}].\end{aligned}$$

Thus we have the following

THEOREM 9. *In a compact Sasakian $M$, the following integral formula is valid for any skew symmetric tensor field $u_{ji}$:*

$$(7.7)\qquad \begin{aligned}\int_M&[u^{it}\{\nabla^j\nabla_j u_{it}+(2n-3)\rho_{it}+\rho_{ti}+\eta^k(2\rho_{kt}\eta_i-\eta_t\rho_{ki}+\eta_t\rho_{ik})\\&-\rho^k(2\varphi_{ki}\eta_t-\varphi_{it}\eta_k+\varphi_{kt}\eta_i)+K_{kit}{}^r u_r{}^k+K_i{}^r u_{rt}\}+\frac{1}{2}A_{jit}A^{jit}\\&-3\rho^t\eta^j\eta^i\nabla_j u_{it}]dV=0\end{aligned}$$

*where $dV$ means the volume element of $M$ and*

$$(2n-1)\rho_{it}=\nabla_i\nabla^r u_{rt}-\frac{1}{2n}(\varphi_i{}^k\nabla^r u_{rk}+\eta^k\nabla_i\nabla^r u_{rk})\eta_t-\frac{1}{2n}(\eta^k\nabla^r u_{rk})\varphi_{it}.$$

Taking account of (7.1), (7.2) and (7.7), we have the following (Yano, [6])

THEOREM 10. *In a compact Sasakian manifold $M$, a necessary and sufficient condition for a skew symmetric tensor $u_{ji}$ to be an $\eta$-conformal Killing tensor is that* (7.1) *and* (7.2).

## 8. $\eta$-conformal Killing tensors in a $C$-Fubinian manifold.

Let $M$ be the integral hypersurface of $X_\lambda dy^\lambda=0$, $X_\lambda$ being the solution covector of (5.13), in a Kaehlerian manifold $'M$ of constant holomorphic sectional curvature. If $M$ is a Sasakian manifold then the curvature tensor $K_{kjih}$ of $M$ is given by (5.19), that is, the $C$-holomorphic sectional curvature of $M$ is equal to 5. We already showed in §5 that if $v_{\mu\lambda}$ is a parallel tensor in $'M$, then $u_{ji}=B_j{}^\mu B_i{}^\lambda v_{\mu\lambda}$ is an $\eta$-conformal Killing tensor in $M$. In this section, we investigate a decomposition of such an $\eta$-conformal Killing tensor $u_{ji}$. In this case from (5.19), we obtain

$$K_{ji}=(2n+2)\gamma_{ji}+2ng_{ji}. \tag{8.1}$$

Substituting (8.1) into (6.16) and taking account of (6.14), we obtain

$$(2n+2)(\eta_j u_{rt}+\eta_t u_{rj})\eta^r =2(n-1)\sigma_{tj}+(\sigma_{rt}\eta_j+\sigma_{rj}\eta_t)\eta^r+3(\varphi_j{}^r\eta_t+\varphi_t{}^r\eta_j)\rho_r. \tag{8.2}$$

Transvecting (8.2) with $\eta^j$ and taking account of (6.14), we get

$$(2n+2)\eta^t u_{tj}=(2n-1)\eta^t\sigma_{tj}+3\varphi_{jt}\rho^t. \tag{8.3}$$

Substituting (8.3) into (8.2), we obtain

$$\sigma_{tj}=\eta^r(\sigma_{rt}\eta_j+\sigma_{rj}\eta_t). \tag{8.4}$$

If we assume $\rho_k=c\eta_k$, $c$ being a non-zero constant, then we easily see that

$$\nabla_j\rho_k+\nabla_k\rho_j=0, \tag{8.5}$$

that is, $\rho^k$ is a Killing vector.

Conversely, if $\rho^k$ is a Killing vector then $\sigma_{kj}=0$ by virtue of (6.8). In this case, we have

$$(2n+2)\eta^t u_{tj}=3\varphi_{jt}\rho^t \tag{8.6}$$

by virtue of (8. 3). Operating $\nabla_k$ to (8. 6) and taking account of (5. 4), we obtain

$$(2n+2)(\varphi_k{}^t u_{tj}+\eta^t \nabla_k u_{tj})=3(-g_{kj}\eta_t\rho^t+\rho_k\eta_j)+3\varphi_{jt}\nabla_k\rho^t.$$

Transvecting this equation with $\eta^k$ and taking account of (5. 20), we obtain $\varphi_{jt}\eta^k\nabla_k\rho^t=0$. Transvecting this equation with $\varphi_i{}^j$ and taking account of the fact that $\rho^k$ is a Killing vector, we see that

$$\eta^k\nabla_t\rho_k=0. \tag{8.7}$$

Operating $\nabla_t$ to $\eta^k\rho_k$ and taking account of (8. 7), we obtain

$$\nabla_t(\eta^k\rho_k)=\varphi_t{}^k\rho_k. \tag{8.8}$$

If $\eta^k\rho_k=0$, then $\varphi_t{}^k\rho_k=0$. Transvecting this equation with $\varphi_i{}^t$, we easily see that $\rho_k=0$. This is contradict to our assumption. Therefore $\eta^k\rho_k\neq 0$. Putting $\lambda=\eta^k\rho_k$ and taking account of (8. 6) and (8. 8), we obtain

$$\eta^t u_{tj}=k\nabla_j\lambda \tag{8.9}$$

$k$ being a constant.

Operating $\nabla_i$ to (8. 9) and taking account of the fact that $\nabla_j\nabla_i\lambda=\nabla_i\nabla_j\lambda$, we obtain

$$\varphi_i{}^t u_{tj}+\eta^t\nabla_i u_{tj}=\varphi_j{}^t u_{ti}+\eta^t\nabla_j u_{ti}.$$

Transvecting this equation with $\eta^j$ and taking account of (5. 20), we obtain $\varphi_i{}^t u_{tj}\eta^j=0$. Transvecting this equation with $\varphi_k{}^i$, we see that $\eta^k u_{kj}=0$ and from which

$$\varphi_{jt}\rho^t=0 \tag{8.10}$$

by virtue of (8. 6).

Transvecting (8. 10) with $\varphi_i{}^j$, we obtain $\rho_i=\lambda\eta_i$. Operating $\nabla_k$ to this equation and substituting it into (8. 5), we see that $(\eta^k\nabla_k\lambda)\eta_i+\nabla_i\lambda=0$ and from which $\nabla_i\lambda=0$. that is, $\lambda=$const. Thus we obtain $\rho_i=c\eta_i$ $c$ being a constant.

Thus we have the following

THEOREM 11. *Let $M$ be a $C$-Fubinian manifold whose $C$-holomorphic sectional curvature is equal to* 5. *Then the associated vector $\rho^k$ of an $\eta$-conformal Killing tensor $u_{ji}$ is a Killing vector if and only if $\rho_t=c\eta_t$ where $\eta_t$ is the* 1-*form of $M$ and $c$ is a non-zero constant.*

We consider an $\eta$-conformal Killing tensor $u_{ji}$ whose associated vector $\rho^k$ is a Killing vector in a $C$-Fubinian manifold whose $C$-holomorphic sectional curvature is equal to 5. In this case, if we put $p_{ji}=u_{ji}+c\varphi_{ji}$ where $c$ is a constant such that $\rho_t = c\eta_t$ then we see that

(8. 11) $$\nabla_k p_{ji} + \nabla_j p_{ki} = 0$$

by virtue of (5. 20) and (5. 21). Thus such an $\eta$-conformal Killing tensor $u_{ji}$ is decomposed in the form

(8. 12) $$u_{ji} = p_{ji} - c\varphi_{ji},$$

where $p_{ji}$ is a Killing tensor by virtue of (8. 11).

The uniqueness of the decomposition (8. 12) is proved by the following way. In fact if

$$u_{ji} = p_{ji} - c\varphi_{ji} = 'p_{ji} - 'c\varphi_{ji},$$

then putting

(8. 13) $$P_{ji} = p_{ji} - 'p_{ji} = (c - c')\varphi_{ji},$$

$P_{ji}$ becomes a closed Killing tensor since $\varphi_{ji}$ is closed and $p_{ji} - 'p_{ji}$ is a Killing tensor. Therefore

$$\nabla_k P_{ji} + \nabla_i P_{kj} + \nabla_j P_{ik} = 0, \qquad \nabla_k P_{ji} + \nabla_j P_{ki} = 0.$$

Hence we get $\nabla_k P_{ji} = 0$. Thus by virtue of Ricci identity it follows that

(8. 14) $$K_{tkj}{}^s P_{si} + K_{tki}{}^s P_{js} = 0.$$

On the other hand, we easily see that

(8. 15) $$P_{ti}\eta^i = 0$$

by virtue of (8. 13).

Transvecting (8. 14) with $\eta^i$ and taking account of (8. 15) and the fact that $K_{kjht}\eta^t = g_{jh}\eta_k - g_{kh}\eta_j$ we obtain $\eta_t P_{jk} - \eta_k P_{jt} = 0$ and from which $P_{jk} = 0$ by virtue of (8. 15), that is, $p_{ji} = 'p_{ji}$ and $c = 'c$. Thus we have the following

THEOREM 12. *Let $M$ be a C-Fubinian manifold whose C-holomorphic sectional curvature is equal to* 5. *Then an $\eta$-conformal Killing tensor $u_{ji}$ whose associated vector $\rho^k$ is a Killing vector in $M$ is decomposed uniquely in the form*

$$u_{ji} = p_{ji} - c\varphi_{ji},$$

*where $p_{ji}$ is a Killing tensor such that $\eta^j p_{ji} = 0$, $\varphi_{ji}$ is the structure tensor in $M$ and $c$ is a non-zero constant.*

## References

1. D.E. Blair, *The theory of quasi-Sasakian structure*, Jour. of Diff. Geometry, **1** (1967), 331-345.
2. K. Ogiue, *On almost contact manifolds admitting axiom of planes or axiom of free mobility*, Kōdai Math. Sem. Rep., **16** (1964), 223-232.
3. S. Sasaki and Y. Hatakeyama, *On differentiable manifolds with metric structures*, Jour. of Math. Soc. of Japan, **14** (1962), 249-271.
4. S. Tachibana, *On conformal Killing tensors in a Riemannian space*, Tōhoku Math. Jour., **21** (1969), 56-64.
5. Y. Tashiro, *On contact structue of hypersurfaces in complex manlfold* I, Tōhoku Math. Jour., **15** (1963), 62-78.
6. Y. Tashiro and S. Tachibana, *On Fubinian and C-Fubinian manifolds*, Kōdia Math. Sem. Rep., **15** (1963), 176-183.
7. K. Yano, *On contact conformal connections*, Kōdai Math. Sem. Rep., **28** (1976), 90-103.
8. K. Yano, S.-S. Eum and U-H. Ki, *On transversal hypersurfaces of an almost contact manifold*, Kōdai Math. Sem. Rep., **24** (1972), 459-470.
9. K. Yano, *Differential geometry on complex and almost complex spaces*, Pergamon Press. 1965.

Sung Kyun Kwan University
Seoul 110, Korea