

TRANSLATION THEOREM FOR CONDITIONAL YEH-WIENER INTEGRALS

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1. Introduction

J. Yeh has recently obtained some useful results for the conditional Wiener integral which is meant the conditional expectation $E^w[Y|X]$ of a real or complex valued Wiener integrable functional Y conditioned by the Wiener measurable functional X on the Wiener measure space (see [8]).

We define the conditional Yeh-Wiener integral on the Yeh-Wiener measure space in the same way that J. Yeh defines the conditional Wiener integral on the Wiener measure space. By a conditional Yeh-Wiener integral we mean the conditional expectation $E^y[Y|X]$ of a real or complex valued Yeh-Wiener integrable functional Y conditioned by the Yeh-Wiener measurable functional X on the Yeh-Wiener measure space $(C_2[Q], \mathcal{U}, m_y)$ defined by

$$(1.1) \quad X(x) = x(p, q) \text{ for } x \in C_2[Q],$$

where $Q = [0, p] \times [0, q]$ is a fixed rectangle in \mathbf{R}^2 .

The conditional expectation $E^y[Y|X]$ is not given as an equivalence class of random variables on the probability space $(C_2[Q], \mathcal{U}, m_y)$, but (as an equivalence class of random variables on the probability space $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1), P_X)$, where P_X is the probability distribution of X defined by

$$(1.2) \quad P_X(B) = m_y(X^{-1}(B)) = \frac{1}{\sqrt{2\pi pq}} \int_B \exp\left(-\frac{u^2}{2pq}\right) dm(u) \text{ for}$$

every $B \in \mathcal{B}(\mathbf{R}^1)$, where $\mathcal{B}(\mathbf{R}^1)$ is the Borel class in \mathbf{R}^1 and m is the Lebesgue measure on $\mathcal{B}(\mathbf{R}^1)$.

We shall use the same notation $E^y[Y|X]$ to mean also the individual representatives of the equivalence class, i. e., the versions of the conditional Yeh-Wiener integral.

In [9], J. Yeh has obtained the translation theorem for the conditional

Wiener integral which is an analogue of the Cameron–Martin Translation Theorem. In this paper, we present the translation theorem for the conditional Yeh–Wiener integral. The proof of this translation theorem is based on the Cameron–Martin Translation Theorem and a translation theorem for the conditional expectation in general. An example of evaluation of conditional Yeh–Wiener integral by means of the translation theorem is given in Section 3.

2. Preliminaries

Let $Q = [0, p] \times [0, q]$ be a fixed rectangle in \mathbf{R}^2 and let $C_2[Q]$ be the Yeh–Wiener space, i. e.,

$$C_2[Q] = \{x(\cdot, \cdot) : x(s, t) \text{ is continuous on } Q, x(0, t) = x(s, 0) = 0\}.$$

Throughout this paper we consider the Yeh–Wiener measure space $(C_2[Q], \mathcal{Y}, m_y)$ as a complete probability space. For a detailed discussion of the Yeh–Wiener measure space consult [2] and [5].

DEFINITION 2.1. Let X and Z be the real valued Yeh–Wiener measurable functionals on the Yeh–Wiener measure space $(C_2[Q], \mathcal{Y}, m_y)$. Suppose Z is Yeh–Wiener integrable. The equivalence class of $\mathcal{B}(\mathbf{R}^1)$ -measurable and P_X -integrable functions f on \mathbf{R}^1 satisfying

$$(2.1) \quad \int_{X^{-1}(B)} Z(x) \, dm_y(x) = \int_B f(u) \, dP_X(u) \text{ for every } B \in \mathcal{B}(\mathbf{R}^1)$$

is called the conditional Yeh–Wiener integral of Z given X and is denoted by $E^y[Z|X]$, the equivalence relation being that of a. e. equality with respect to P_X . Each member of this equivalence class is called a version (or representative) of the conditional Yeh–Wiener integral.

REMARK. From the Radon–Nikodym Theorem it follows that a function f as in (2.1) always exists and is determined uniquely up to a null set of $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1), P_X)$. We shall use $E^y[Z|X]$ to mean either the equivalence class of all functions f as in (2.1) or a version in it depending on the context. Thus we have

$$(2.2) \quad \int_{X^{-1}(B)} Z(x) \, dm_y(x) = \int_B E^y[Z|X](u) \, dP_X(u) \text{ for every } B \in \mathcal{B}(\mathbf{R}^1).$$

DEFINITION 2.2. A real valued function $f(s, t)$ is said to be of bounded variation on Q if the following are satisfied:

- (1) $f(s, 0)$ is of bounded variation on $[0, p]$
- (2) $f(0, t)$ is of bounded variation on $[0, q]$
- (3) The total variation V_f of $f(s, t)$ on Q is finite, where V_f is the

supremum of

$$\sum_{i,j=1}^{m,n} |f(s_i, t_j) - f(s_{i-1}, t_j) - f(s_i, t_{j-1}) + f(s_{i-1}, t_{j-1})|$$

for any subdivision: $0=s_0 < s_1 < \dots < s_m=p$, $0=t_0 < t_1 < \dots < t_n=q$ with m and n arbitrary.

DEFINITION 2.3. Let $L^2[Q]$ be the real Hilbert space of all square integrable functions with respect to the Lebesgue measure m on Q . Let $\{\varphi_k(s, t) : k=1, 2, 3, \dots\}$ be a complete orthonormal set (in brief, C. O. N set) on $L^2[Q]$, where each $\varphi_k(s, t)$ is of bounded variation on Q , $k=1, 2, 3, \dots$. Then the Paley-Wiener-Zygmund integral of $f(s, t) \in L^2[Q]$ with respect to $x(s, t)$ continuous on Q is defined by

$$(2.3) \quad \int_Q f(s, t) \hat{d}x(s, t) = \lim_{n \rightarrow \infty} \int_Q \left[\sum_{k=1}^n c_k \varphi_k(s, t) \right] dx(s, t)$$

if this limit exists, where $c_k = \int_Q f(s, t) \varphi_k(s, t) dm(s, t)$, $k=1, 2, \dots, n$.

For the detailed discussion of bounded variation for functions with two variables see [2]. It has been shown by C. Park [3] that the limit defining the Paley-Wiener-Zygmund integral exists for almost all $x \in C_2[Q]$ and that this limit is essentially independent of the choice of the C. O. N set $\{\varphi_k(s, t) : k=1, 2, 3, \dots\}$; further, if $f(s, t)$ is of bounded variation, then the Paley-Wiener-Zygmund integral $\int_Q f(s, t) \hat{d}x(s, t)$ is m_y -a. e. equal to the Riemann-Stieltjes integral $\int_Q f(s, t) dx(s, t)$.

The following two propositions which are used to prove Proposition 3-1 in the next section are taken from [1, Proposition 5-5] and [1, Theorem 5-6], respectively.

PROPOSITION 2.1. Let $\{\varphi_k(s, t) : k=1, 2, 3, \dots\}$ be a C. O. N. set as in Definition 2-3, and define a random variable X_k by

$$X_k(x) = \int_Q \varphi_k(s, t) dx(s, t) \text{ for every } x \in C_2[Q],$$

where $k=1, 2, 3, \dots$, and $\int_Q \varphi_k(s, t) dx(s, t)$ is the ordinary Riemann-Stieltjes integral. Then $\{X_k : k=1, 2, 3, \dots\}$ is a sequence of independent random variables such that each X_k is normally distributed with mean 0 and variance 1.

PROPOSITION 2.2. Let $\{\varphi_k(s, t) : k=1, 2, 3, \dots\}$ be a C. O. N set as in Definition 2-3, and let $f \in L^2[Q]$. Define a functional Y_n on $C_2[Q]$ as follows:

$Y_n(x) = \int_Q \left[\sum_{k=1}^n c_k \varphi_k(s, t) \right] dx(s, t)$ for $x \in C_2[Q]$, where $n=1, 2, 3, \dots$, and $c_k = \int_Q f(s, t) \varphi_k(s, t) dm(s, t)$, $k=1, 2, \dots, n$. Then the sequence $\{Y_n : n=1, 2, 3, \dots\}$ converges in the $L^2[C_2[Q]]$ mean to $\int_Q f(s, t) \hat{d}x(s, t)$.

The following two theorems which are needed in the next section are taken from [3; Theorem 1.4] and [9; Theorem 1], respectively, and we state them in a convenient form for our purposes without proof.

THEOREM 2.1. (*Generalized Cameron–Martin Translation Theorem*) Let $f \in L^2[Q]$ and let $x_0(s, t) = \int_0^t \int_0^s f(u, v) dm(u, v)$ for $(s, t) \in Q$, where $Q = [0, p] \times [0, q]$. Define a translation T from the Yeh–Wiener space $C_2[Q]$ into itself by $T(x) = x + x_0$ for $x \in C_2[Q]$. Then we have the following results:

(I) If M is an Yeh–Wiener measurable set, then $T(M) = M + x_0$ is also an Yeh–Wiener measurable set, and

$$(2.4) \quad m_y[T(M)] = \int_M J(x) dm_y(x),$$

where $J(x) = \exp\left\{-\frac{1}{2} \int_Q f^2(s, t) dm(s, t)\right\} \cdot \exp\left\{-\int_Q f(s, t) \hat{d}x(s, t)\right\}$.

(II) If $F(x)$ is an Yeh–Wiener measurable functional, then $F(x + x_0)$ is also Yeh–Wiener measurable, and

$$(2.5) \quad \int_{C_2[Q]} F(x) dm_y(x) \stackrel{*}{=} \int_{C_2[Q]} J(x) F(x + x_0) dm_y(x),$$

where $J(x)$ is as in (2.4), and $\stackrel{*}{=}$ means that the existence of either integral implies the existence of the other and their equality.

THEOREM 2.2. (*Transformation Theorem for conditional expectation*) Given a real valued integrable random variable Z on a probability space (Ω, \mathcal{B}, P) and a measurable transformation X from (Ω, \mathcal{B}) into a measurable space (S, \mathcal{F}) . Let T be a measurable transformation from (Ω, \mathcal{B}) into itself which satisfies the following three conditions:

(1) There exists a real valued \mathcal{B} -measurable function J on Ω such that

$$P(B) = \int_{T^{-1}(B)} J(w) dP(w) \text{ for every } B \in \mathcal{B}.$$

(2) There exists a one to one transformation h from S onto itself such that both h and h^{-1} are measurable transformations from (S, \mathcal{F}) into itself and further,

$(X \cdot T)(w) = (h \cdot X)(w)$ for a. e. w in (Ω, \mathcal{B}, P) .

(3) $P_{X \cdot T}$ is absolutely continuous with respect to P_X on (S, \mathcal{F}) . Then for arbitrary versions of two conditional expectations $E[Z|X]$ and $E[(Z \cdot T)J|X]$, we have

$$(2.6) \quad E[Z|X](u) = E[(Z \cdot T)J|X](h^{-1}(u)) \frac{dP_{X \cdot T}}{dP_X}(u) \text{ for a. e. } u \text{ in } (S, \mathcal{F}, P_X).$$

3. Translation of conditional Yeh-Wiener integrals

In this section, we shall state and prove our main results and give an example of evaluation of conditional Yeh-Wiener integral by means of the translation theorem.

First of all we need the following two propositions. In [1], he has shown that Theorem 2-1 implies Proposition 3-2 from which Proposition 3-1 is derived. It turns out that Proposition 3-1 can be proved directly without the help of Theorem 2-1 and then it implies Proposition 3-2. It is worth showing the proofs.

PROPOSITION 3.1. Let $f \in L^2[Q]$ and let

$$X_f(x) = \int_Q f(s, t) \hat{d}x(s, t) \text{ for every } x \in C_2[Q].$$

Then we have

(I) X_f is Yeh-Wiener measurable and X_f is normally distributed with mean 0 and variance $\|f\|_2^2$.

(II) If g is any Lebesgue measurable function on \mathbf{R}^1 , then

$$\begin{aligned} & \int_{C_2[Q]} g \left[\int_Q f(s, t) \hat{d}x(s, t) \right] dm_y(x) \\ & \stackrel{*}{=} \frac{1}{\sqrt{2\pi}\|f\|_2} \int_{\mathbf{R}^1} g(u) \exp \left\{ -\frac{u^2}{2\|f\|_2^2} \right\} dm(u), \end{aligned}$$

where $*$ has the same meaning as in (2, 5).

Proof. Let $f \in L^2[Q]$ and let $\{\varphi_k(s, t) : k=1, 2, 3, \dots\}$ be a C. O. N set for $L^2[Q]$ as in Definition 2-3. Define

$$Y_n(z) = \int_Q \left[\sum_{k=1}^n c_k \varphi_k(s, t) \right] dx(s, t) \text{ for every } x \in C_2[Q],$$

where $c_k = \int_Q f(s, t) \varphi_k(s, t) dm(s, t), \quad k=1, 2, \dots, n.$

Then by Proposition 2-2, the sequence $\{Y_n : n=1, 2, 3, \dots\}$ converges in the

$L^2[C_2[Q]]$ mean to X_f and by Proposition 2-1, each Y_n is normally distributed with mean 0 and variance $\sum_{k=1}^n c_k^2$. Hence by [4, Theorem 4B, p. 91], X_f is normally distributed with mean 0 and variance $\|f\|_2^2$. Thus (I) is proved.

Since X_f is normally distributed and m_y is complete, $g \cdot X_f$ is Yeh-Wiener measurable and by Change of Variable Theorem, (II) is proved.

PROPOSITION 3. 2. *Let $f \in L^2[Q]$. Then for every complex number λ , we have*

$$(3. 0) \quad \int_{C_2[Q]} \exp \left\{ \lambda \int_Q f(s, t) \hat{d}x(s, t) \right\} dm_y(x) = \exp \left\{ \frac{\lambda^2}{2} \|f\|_2^2 \right\}.$$

Proof. If λ is real, then (3. 0) follows immediately from Proposition 3-1 (II) by setting $g(u) = \exp(\lambda u)$, $u \in \mathbf{R}^1$. If λ is complex, then we can prove this proposition by the same way as in the proof of Corollary in [6, p. 1429].

REMARK. Formula (3. 0) is well known for the functions f which are of bounded variation on Q (see [2] and [6]).

THEOREM 3. 1. *Let Y be a real valued Yeh-Wiener integrable functional on the Yeh-Wiener measure space $(C_2[Q], \mathcal{U}, m_y)$. Define $X(x) = x(p, q)$ for $x \in C_2[Q]$, and let $x_0(s, t) = \int_0^t \int_0^s f(u, v) dm(u, v)$ for $(s, t) \in Q$, where $f \in L^2[Q]$. Then for arbitrary versions of the conditional Yeh-Wiener integrals $E^y[Y|X]$ and $E^y[Y(\cdot + x_0)J|X]$, we have*

$$(3. 1) \quad E^y[Y|X](w) = E^y[Y(\cdot + x_0)J|X](w - x_0(p, q)) \exp \left[-\frac{1}{2pq} \left\{ [x_0(p, q)]^2 - 2wx_0(p, q) \right\} \right]$$

for m -a. e. w in \mathbf{R}^1 , where

$$(3. 2) \quad J(X) = \exp \left\{ -\frac{1}{2} \int_Q f^2(s, t) ds dt \right\} \cdot \exp \left\{ -\int_Q f(s, t) \hat{d}x(s, t) \right\}$$

for m_y -a. e. x in $C_2[Q]$.

Proof. Let T be the transformation from $C_2[Q]$ into itself defined by

$$T(x) = x + x_0 \text{ for } x \in C_2[Q].$$

Then it is obvious that the condition (1) in Theorem 2-2 is satisfied by T from Theorem 2-1.

Now let us show that T satisfies the condition (2) in Theorem 2-2. With T , we have

$$(X \cdot T)(x) = X(x + x_0) = x(p, q) + x_0(p, q) = (h \cdot X)(x) \text{ for}$$

$x \in C_2[Q]$, where h is the one-to-one transformation from \mathbf{R}^1 onto itself defined by

$$h(w) = w + x_0(p, q) \text{ for } w \in \mathbf{R}^1$$

for which we have

$$h^{-1}(w) = w - x_0(p, q) \text{ for } w \in \mathbf{R}^1.$$

Clearly both h and h^{-1} are measurable transformations of $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$ into itself. Thus T satisfies the condition (2) in Theorem 2-2, where (S, \mathcal{F}) is now $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$.

Finally, to verify the condition (3) of Theorem 2-2, recall that the random variable $X(x) = x(p, q)$ is normally distributed with mean 0 and variance pq , and the random variable $(X \cdot T)(x) = x(p, q) + x_0(p, q)$ is normally distributed with mean $x_0(p, q)$ and variance pq , where x varies in $C_2[Q]$. Thus $P_{X \cdot T}$ is absolutely continuous with respect to P_X on $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$ and as a version of the Radon-Nikodym derivative of $P_{X \cdot T}$ with respect to P_X we have

$$\begin{aligned} (3.3) \quad \frac{dP_{X \cdot T}}{dP_X}(w) &= \left[\frac{dP_{X \cdot T}}{dm}(w) \right] \cdot \left[\frac{dP_X}{dm}(w) \right]^{-1} \\ &= \exp \left\{ -\frac{[x_0(p, q)]^2}{2pq} \right\} \cdot \exp \left\{ \frac{wx_0(p, q)}{pq} \right\} \text{ for } w \in \mathbf{R}^1. \end{aligned}$$

On the other hand, setting $(\Omega, \mathcal{B}, P) = (C_2[Q], \mathcal{U}, m_y)$, we rewrite (2.6) as follows:

$$(3.4) \quad E^y[Y|X](w) = E^y[(Y \cdot T)J|X](h^{-1}(w)) \frac{dP_{X \cdot T}}{dP_X}(w) \text{ for } P_X\text{-a. e. } w.$$

Combining (3.3) and (3.4), we have the desired result since P_X and m are equivalent on $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$.

THEOREM 3.2. *Let $X(x)$ and x_0 be as in Theorem 3-1. Then one version of the conditional Yeh-Wiener integral $E^y[Z|X]$, where $Z(x) = \exp \left\{ -\int_Q f(s, t) \hat{d}x(s, t) \right\}$ for $x \in C_2[Q]$, and $f \in L^2[Q]$, is given by*

$$\begin{aligned} (3.5) \quad E^y[Z|X](w) &= \exp \left\{ \frac{1}{2} \int_Q f^2(s, t) ds dt \right\} \cdot \\ &\quad \exp \left[-\frac{1}{2pq} \left\{ (x_0(p, q))^2 + 2wx_0(p, q) \right\} \right] \\ &\text{for } w \in (\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1), m). \end{aligned}$$

Proof. By (3.0), the random variable Z is Yeh-Wiener integrable. Hence $E^y[Z|X]$ exists. Let $Y \equiv 1$ on $C_2[Q]$. Then one version of the con-

ditional Yeh-Wiener integral $E^y[Y|X]$ is given by

$$(3.6) \quad E^y[Y|X](w) = 1 \text{ for } P_X\text{-a. e. } w.$$

Let J be as given by (3.2). Since $Y(x+x_0)=1$ for $x \in C_2[Q]$, we have

$$(3.7) \quad E^y[Y(\cdot + x_0)J|X](w) = E^y[J|X](w) \\ = \exp\left\{-\frac{1}{2}\int_Q f^2(s, t) \, ds \, dt\right\} \cdot E^y[Z|X](w) \text{ for } P_X\text{-a. e. } w$$

for arbitrary versions of the two conditional Yeh-Wiener integrals involved.

Since P_X and m are equivalent on $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1))$, the equality (3.7) holds for m -a. e. w . Since a translate of a null set in $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1), m)$ is again a null set in $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1), m)$, if we replace w in both sides of (3.7) by $u - x_0(p, q)$, then the equality holds for m -a. e. u , i. e.,

$$E^y[Y(\cdot + x_0)J|X](u - x_0(p, q)) = \exp\left\{-\frac{1}{2}\int_Q f^2(s, t) \, ds \, dt\right\} \\ \cdot E^y[Z|X](u - x_0(p, q))$$

for m -a. e. u .

Using this equality and (3.6) in the equality (3.1), we have

$$(3.8) \quad 1 = E^y[Z|X](u - x_0(p, q)) \cdot \exp\left\{-\frac{1}{2}\int_Q f^2(s, t) \, ds \, dt\right\} \\ \cdot \exp\left[-\frac{1}{2pq}\left\{[x_0(p, q)]^2 - 2ux_0(p, q)\right\}\right]$$

for m -a. e. u , for an arbitrary version of the conditional Yeh-Wiener integral involved. Putting $u = w + x_0(p, q)$ and recalling that a translate of a null set in $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1), m)$ is again a null set in $(\mathbf{R}^1, \mathcal{B}(\mathbf{R}^1), m)$, we have the desired result from (3.8).

The following example is an application of Theorem 3-2 in evaluating a conditional Yeh-Wiener integral.

EXAMPLE. Let

$$(3.9) \quad X(x) = x(p, q) \text{ and } Y(x) = \exp\left\{\lambda \int_Q h(s, t) \, dx(s, t)\right\}$$

for every $x \in C_2[Q]$, where $\lambda \in \mathbf{R}^1$ and $h \in L^2[Q]$.

Define an element x_0 in $C_2[Q]$ by

$$x_0(s, t) = -\lambda \int_0^t \int_0^s h(u, v) \, du \, dv \text{ for } (s, t) \text{ in } Q.$$

Then $-\lambda h(u, v)$ is as $f(u, v)$ in Theorem 3-2.

With x_0 , Y can be written as

$$Y(x) = \exp\left\{-\int_Q f(s, t) \hat{d}x(s, t)\right\} \text{ for } x \in C_2[Q].$$

Thus, by (3.5), a version of $E^y[Y|X]$ is given by

$$(3.10) \quad E^y[Y|X](w) = \exp\left\{\frac{\lambda^2}{2} \int_Q h^2(s, t) \, ds \, dt\right\} \\ \cdot \exp\left[-\frac{\lambda^2}{2pq} \left\{\int_Q h(s, t) \, ds \, dt\right\}^2\right] \\ \cdot \exp\left\{\frac{w\lambda}{pq} \int_Q h(s, t) \, ds \, dt\right\} \text{ for } w \in \mathbf{R}^1.$$

Therefore, for every B in $\mathcal{B}(\mathbf{R}^1)$, we have

$$(3.11) \quad \int_{X^{-1}(B)} \exp\left\{\lambda \int_Q h(s, t) \, \hat{d}x(s, t)\right\} dm_y(x) \\ = \int_B E^y[Y|X](w) \, dP_X(w) \\ = \exp\left\{\frac{\lambda^2}{2} \int_Q h^2(s, t) \, ds \, dt\right\} \\ \cdot \frac{1}{\sqrt{2\pi pq}} \int_B \exp\left[-\frac{1}{2pq} \left\{w - \lambda \int_Q h(s, t) \, ds \, dt\right\}^2\right] dm(w)$$

by (3.10) and the fact that X is normally distributed with mean 0 and variance pq .

In particular, if we substitute \mathbf{R}^1 for B in both sides of (3.11), then we have

$$\int_{C_2[Q]} \exp\left\{\lambda \int_Q h(s, t) \, \hat{d}x(s, t)\right\} dm_y(x) = \exp\left\{\frac{\lambda^2}{2} \int_Q h^2(s, t) \, ds \, dt\right\},$$

which is the same result as in Proposition 3-2 for $\lambda \in \mathbf{R}^1$. This last equality is also obtained by the second Yeh-Wiener integration formula. (For the second Yeh-Wiener integration formula see Theorem 1.5 in [3, p.12].).

Acknowledgements: The first author acknowledges the support of the Korea Science Foundation and Yonsei University. In addition he acknowledges the support and hospitality of the Department of Mathematics, University of Nebraska and especially the kind help of Professors G. W. Johnson and D. L. Skoug.

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