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GENERIC SUBMANIFOLDS WITH NORMAL STRUCTURE OF AN ODD-DIMENSIONAL SPHERE (I)

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0. Introduction

Recently, generic submanifolds of a Kaehlerian manifold-mainly a complex projective space-have been studied by several authors who used the method of Riemannian fibre bundles (see [5], [6], [7] and [10] etc.).

In the present paper, we consider the so-called generic submanifolds of a Sasakian manifold. The purpose of this paper is to study generic submanifolds of an odd-dimensional sphere S^{2m+1} (1) with radius 1 under the condition that the structure tensor induced on the submanifold is normal (see §1). We will write S^{2m+1} instead of $S^{2m+1}(1)$ to simplify the notation throughout the paper.

In determining the submanifolds, we will make use of the following theorem;

THEOREM A (K. Yano and M. Kon [8]). *Let M be a complete n-dimensional submanifold of a sphere S^m of dimension m with flat normal connection. If the second fundamental form of M is parallel, then M is a small sphere, a great sphere or a pythagorean product of a certain number of spheres. Moreover, if M is of essential codimension $m-n$. then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),\quad r_1^2+\cdots+r_N^2=1,\quad N=m-n+1,$$

or a pythagorean product of the form

$$S^{p_1}(r_1)\times\cdots\times S^{p_{N'}}(r_{N'})\subset S^{m-1}(r)\subset S^m,\quad r_1^2+\cdots+r_{N'}^2=r^2<1,\quad N'=m-n.$$

THEOREM B (Yano and Ishihara [9]). *Let M be a complete minimal submanifold of dimension n immersed in an m-dimensional sphere $S^m(a)$ with radius a $(0<a<1,\ 1<n<m)$ and nonnegative sectional curvature, and suppose the normal bundle $N(M)$ of M is locally parallelizable. If the function $F=h_{cb}{}^x h^{cb}{}_x$ is constant in M, then M is a great sphere of $S^m(a)$ or a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),\quad p_1,\cdots,p_N\geqq 1,\quad p_1+\cdots+p_N=n,\quad 1<N\leqq m-n+1$$

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with essential codimension $N-1$, *where* $r_\alpha=a\sqrt{p_\alpha/n}$ $(\alpha=1, \ldots, N)$.

By the way the sectional curvature of n-dimensional submanifold immersed in S^m with flat normal connection is always nonnegative if the second fundamental tensor of the submanifold is parallel (for detail see [7]). Hence we have

REMARK A. Let M be a complete minimal submanifold of dimension n immersed in an m-dimensional sphere S^m $(1<n<m)$ with flat normal connection. If the second fundamental tensor of M is parallel, then M is the same type as stated in Theorem B.

1. Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and $(F_j{}^h, g_{ji}, F_i)$ the set of structure tensors of $\tilde{M}$, where, here and in the sequel, the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, 2m+1\}$. Then we have

$$F_i{}^t F_t{}^h = -\delta_i^h + F_i F^h, \quad F_t F_i{}^t = 0, \quad F_t{}^h F^t = 0, \quad F_t F^t = 1 \tag{1.1}$$

and

$$F_j{}^t F_i{}^s g_{ts} = g_{ji} - F_j F_i, \tag{1.2}$$

F^h being the vector field associated with F_i, that is, $F^h = F_i g^{ih}$, g^{ih} being the contravariant metric tensor of $\tilde{M}$. We also have

$$\nabla_j F^h = F_j{}^h \tag{1.3}$$

and

$$\nabla_j F_i{}^h = -g_{ji} F^h + \delta_j^h F_i, \tag{1.4}$$

where ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols formed with g_{ji}.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and isometrically immersed in $\tilde{M}$ by the immersion $i: M \to \tilde{M}$, where, here and in the sequel, the indices $a, b, c, \ldots$ run over the range $\{1, 2, \ldots, n\}$. We identify $i(M)$ with M and represent the immersion by $x^h = x^h(y^a)$. If we put $B_b{}^h = \partial_b x^h$ $(\partial_b = \partial/\partial y^b)$, then $B_b{}^h$ are n linearly independent vectors of $\tilde{M}$ tangent to M. Denoting by g_{cb} the Riemannian metric tensor of M, we have $g_{cb} = g_{ji} B_c{}^j B_b{}^i$ since the immersion is isometric. We denote by $C_x{}^h$ $2m+1-n$ mutually orthogonal unit normals to M, then we have $g_{ji} B_c{}^j C_x{}^i = 0$ and the metric tensor of the normal bundle of M is given by $g_{zy} = g_{ji} C_z{}^j C_y{}^i = \delta_{zy}$, δ_{zy} being the Kronecker delta, where, here and in the sequel, the indices u, v, w, x, y, z run over the range $\{1^*, 2^*, \ldots, p^*\}$, where $n+p=2m+1$. Therefore, denoting by ∇_b the operator of van der Waerden-Bortolotti covariant differentiation with respect to the Christoffel symbols $\left\{ {a \atop c\ b} \right\}$ formed with g_{cb}, we have equations of Gauss and Weingarten for M

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h, \tag{1.5}$$

$$\nabla_c C_x{}^h = -h_c{}^a{}_x B_a{}^h \tag{1.6}$$

respectively, where $h_{cb}{}^x$ are the second fundamental tensors with respect to the normals $C_x{}^h$ and $h_c{}^a{}_x = h_{cb}{}^y g^{ba} g_{yx}$, $(g^{ba}) = (g_{ba})^{-1}$.

Denoting by $K_{kji}{}^h$, $K_{dcb}{}^a$ and $K_{dcy}{}^x$ the curvature tensors of $\tilde{M}$, M and the normal connection of M respectively, we have the following equations of Gauss, Codazzi and Ricci respectively

$$K_{dcb}{}^a = K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i B^a{}_h + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x, \tag{1.7}$$

$$0 = K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i C^x{}_h - (\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x), \tag{1.8}$$

$$K_{dcy}{}^x = K_{kji}{}^h B_d{}^k B_c{}^j C_y{}^i C^x{}_h + h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y, \tag{1.9}$$

where we have put $B^a{}_h = B_b{}^j g^{ba} g_{jh}$, $C^x{}_h = C_y{}^j g^{yx} g_{jh}$ and $(g^{yx}) = (g_{yx})^{-1}$.

An n-dimensional submanifold M is called a *generic* (an *anti-holomorphic*) submanifold of the Sasakian manifold M if M satisfies

$$N_p(M) \perp F(N_p(M))$$

at each point $p \in M$, where $N_p(M)$ denotes the normal space at p.

From now on, we consider generic submanifolds immersed in a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$. Then we can put in each coordinate neighborhood

$$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h - f_b{}^x C_x{}^h, \tag{1.10}$$

$$F_i{}^h C_x{}^i = f_x{}^a B_a{}^h, \tag{1.11}$$

where $f_b{}^a$ is a tensor field of type (1, 1) defined on M, $f_c{}^x$ a local 1-form for each fixed index x and $f_x{}^a = f_c{}^y g^{ca} g_{yx}$. Also, we can put the Sasakian structure vector F^h of the form

$$F^h = f^a B_a{}^h + f^x C_x{}^h, \tag{1.12}$$

f^a and f^x being vector fields defined on M and the normal bundle of M respectively.

Transvecting (1.10) and (1.11) with $F_h{}^k$ respectively and using (1.1), (1.10), (1.11) and (1.12), we find

$$f_b{}^e f_e{}^a = -\delta_b^a + f_b f^a + f_b{}^x f_x{}^a, \tag{1.13}$$

$$f_x{}^e f_e{}^y = \delta_x^y - f_x f^y, \tag{1.14}$$

$$f_b{}^e f_e{}^x = -f_b f^x, \quad f_x{}^e f_e{}^a = f_x f^a, \tag{1.15}$$

where f_c and f_x are 1-forms associated with f^a and f^x respectively, that is, $f_c=f^a g_{ac}$ and $f_x=f^y g_{yx}$.

Also, transvecting (1.12) with $F_h{}^k$ and using (1.1), (1.10) and (1.11), we get

(1.16) $$f_e{}^a f^e+f_x{}^a f^x=0,$$

(1.17) $$f_e{}^x f^e=0.$$

From (1.12), we have

(1.18) $$f_a f^a+f_x f^x=1$$

with the help of $F_t F^t=1$.

If we put $f_{cb}=f_c{}^a g_{ba}$, $f_{cx}=f_c{}^y g_{yx}$ and $f_{xc}=f_x{}^a g_{ca}$, then we can easily verify that

$$f_{cb}=-f_{bc},\ f_{xc}=f_{cx}.$$

If we apply the operator ∇_c of covariant differentiation to (1.10) and take account of (1.4), (1.5) and (1.6), then we find

$$(-g_{ji}F^h+\delta_j{}^h F_i)B_c{}^j B_b{}^i+h_{cb}{}^x f_x{}^a B_a{}^h=(\nabla_c f_b{}^a)B_a{}^h+h_{ca}{}^x f_b{}^a C_x{}^h$$
$$-(\nabla_c f_b{}^x)C_x{}^h+h_c{}^a{}_x f_b{}^x B_a{}^h,$$

which implies

(1.19) $$\nabla_c f_b{}^a=-g_{cb}f^a+\delta_c{}^a f_b+h_{cb}{}^x f_x{}^a-h_c{}^a{}_x f_b{}^x,$$

(1.20) $$\nabla_c f_b{}^x=g_{cb}f^x+h_{ce}{}^x f_b{}^e.$$

By the same way we can also obtain from (1.11),

(1.21) $$\nabla_c f_x{}^a=\delta_c{}^a f_x-h_c{}^e{}_x f_e{}^a,$$

(1.22) $$h_c{}^e{}_x f_e{}^y=h_{ce}{}^y f_x{}^e.$$

Differentiating (1.12) covariantly and using (1.3), we find

$$F_j{}^h B_c{}^j=(\nabla_c f^a)B_a{}^h+h_{ca}{}^x f^a C_x{}^h+(\nabla_c f^x)C_x{}^h-h_c{}^a{}_x f^x B_a{}^h,$$

from which, taking account of (1.10),

(1.23) $$\nabla_c f^a=f_c{}^a+h_c{}^a{}_x f^x,$$

(1.24) $$\nabla_c f^x=-f_c{}^x-h_{ce}{}^x f^e.$$

From (1.13), (1.14) and (1.18) we have

(1.25) $$f_{cb}f^{cb}+2f_c f^c=2(n-m).$$

Thus, if $n=m$, then we get

(1.26) $$f_c=0,\ f_{cb}=0.$$

Conversely, if $f_c=0$, then we see from (1.23) that $f_{cb}=0$ and $h_{cbx}f^x=0$

because f_{cb} is skew-symmetric and $h_{cbx}f^x$ is symmetric. Hence, (1.25) implies $n=m$.

Therefore, we have

REMARK B. For an n-dimensional generic submanifold M of a $(2m+1)$-dimensional Sasakian manifold $\tilde{M}$, $f_c=0$ if and only if $n=m$.

LEMMA 1.1. *Let M be an m-dimensional generic submanifold of a* $(2m+1)$-*dimensional Sasakian manifold. Then we have*

$$h_{cb}{}^x=P_{yz}{}^x f_c{}^y f_b{}^z, \tag{1.27}$$

where we have put

$$P_{yz}{}^x=h_{cb}{}^x f_y{}^c f_z{}^b. \tag{1.28}$$

Proof. Since $m=n$, (1.26) is valid. Thus (1.13) reduces to

$$f_b{}^x f_x{}^a=\delta_b{}^a. \tag{1.29}$$

Transvecting (1.28) with $f_a{}^y f_d{}^z$ and taking account of (1.29), we find

$$P_{yz}{}^x f_a{}^y f_d{}^z=h_{da}{}^x.$$

This proves (1.27).

The mean curvature vector of M is given by $H^h=\frac{1}{n}h^x C_x{}^h$ is said to be parallel in the normal bundle if $\nabla_c h^x=0$, where $h^x=g^{cb}h_{cb}{}^x$.

The induced structure $(f_c{}^a, g_{cb}, f_c{}^x, f_c, f^x)$ satisfying (1.13)$\sim$(1.18) is said to be *normal* if

$$h_c{}^e{}_x f_e{}^a-f_c{}^e h_e{}^a{}_x=0,$$

or, equivalently

$$h_{ce}{}^x f_a{}^e+h_{ae}{}^x f_c{}^e=0. \tag{1.30}$$

Transvection $f^c f^a$ gives

$$h_{ce}{}^x f_a{}^e f^c f^a=-h_{ce}{}^x f^c f_y{}^e f^y=0 \tag{1.31}$$

with the help of (1.16).

Transvecting (1.30) with $f_b{}^a$ and using (1.13), we find

$$h_{ce}{}^x(-\delta_b{}^e+f_b f^e+f_b{}^y f_y{}^e)+h_{ae}{}^x f_c{}^e f_b{}^a=0,$$

that is,

$$h_{cb}{}^x-(h_{ce}{}^x f_y{}^e)f_b{}^y-(h_{ce}{}^x f^e)f_b-h_{ae}{}^x f_c{}^e f_b{}^a=0, \tag{1.32}$$

from which, taking the skew-symmetric part with respect to the indices c and b,

$$(1.33)\qquad (h_{be}{}^x f_y{}^e) f_c{}^y - (h_{ce}{}^x f_y{}^e) f_b{}^y + (h_{be}{}^x f^e) f_c - (h_{ce}{}^x f^e) f_b = 0.$$

We now prove

LEMMA 1.2. *Let M be an n-dimensional generic submanifold of a* $(2m+1)$*-dimensional Sasakian manifold with* $m \neq n$. *If the induced structure on M is normal and the function* $f_x f^x$ *is nonzero almost everywhere on M, then we have*

$$(1.34)\qquad h_{ce}{}^x f^e = \lambda^x f_c,$$

$$(1.35)\qquad h_{ce}{}^x f_y{}^e = Q_{yz}{}^x f_c{}^z,$$

where we have put

$$\lambda^x = h_{de}{}^x f^d f^e / 1 - f^y f_y$$

and

$$(1.36)\qquad Q_{yz}{}^x = \lambda^x f_y f_z + h_{de}{}^x f_y{}^d f_z{}^e.$$

Proof. Transvecting (1.33) with f^b and using (1.17) and (1.18), we find

$$(1.37)\qquad (h_{be}{}^x f^b f_y{}^e) f_c{}^y + (h_{be}{}^x f^b f^e) f_c - (1 - f_y f^y) h_{ce}{}^x f^e = 0,$$

from which, transvecting $f_z{}^c$ and taking account of (1.14),

$$(1.38)\qquad (f_y f^y) h_{ce}{}^x f^e f_z{}^c - (h_{be}{}^x f^b f_y{}^e f^y) f_z = 0.$$

Consequently, we have $h_{ce}{}^x f^e f_z{}^c = 0$ because of (1.31) and the fact that $f_x f^x$ is nonzero almost everywhere. Since $m \neq n$, we have $1 - f_x f^x \neq 0$ on M because of (1.18) and Remark B. Thus, (1.37) reduces to

$$h_{ce}{}^x f^e = \lambda^x f_c$$

and hence (1.33) also to

$$(h_{ce}{}^x f_y{}^e) f_b{}^y - (h_{be}{}^x f_y{}^e) f_c{}^y = 0.$$

If we transvect the last equation with $f_z{}^b$ and taking account of (1.14), we obtain

$$h_{ce}{}^x f_z{}^e - (h_{ce}{}^x f_y{}^e f^y) f_z - h_{de}{}^x f_y{}^e f_c{}^y f_z{}^d = 0,$$

or, using (1.16) and (1.30),

$$h_{ce}{}^x f_z{}^e - (h_{ae}{}^x f^a f_c{}^e) f_z - h_{de}{}^x f_y{}^e f_c{}^y f_z{}^d = 0.$$

Thus, using (1.16) and (1.34), we have

$$h_{ce}{}^x f_z{}^e = (\lambda^x f_y f_z + h_{de}{}^x f_y{}^d f_z{}^e) f_c{}^y,$$

which proves (1.35). Thus Lemma 1.2 is proved.

Putting $Q_{yzx}=Q_{yz}{}^{w}g_{wx}$, we can easily verify that Q_{yzx} is symmetric for any index because of (1.22) and the definition of (1.36).

2. Generic submanifolds of S^{2m+1} with nonzero normal part of its Sasakian structure vector

In this section, we consider that a generic submanifold M of an odd-dimensional unit sphere S^{2m+1}. Then, the equations of Gauss, Codazzi and Ricci for M are respectively given by

(2.1) $$K_{dcb}{}^{a}=\delta_{d}{}^{a}g_{cb}-\delta_{c}{}^{a}g_{db}+h_{d}{}^{a}{}_{x}h_{cb}{}^{x}-h_{c}{}^{a}{}_{x}h_{db}{}^{x},$$

(2.2) $$\nabla_{d}h_{cb}{}^{x}-\nabla_{c}h_{db}{}^{x}=0,$$

since the ambient manifold S^{2m+1} is a space of constant curvature 1.

We first prove

LEMMA 2.1. *Let M be an n-dimensional generic submanifold with flat normal connection of an odd-dimensional unit sphere S^{2m+1}. If the induced structure on M is normal and the function $f_{x}f^{x}$ is nonzero almost everywhere, then M is totally umbilical if the codimension $p=1$ and M has the second fundamental tensors of the form*

(2.4) $$h_{cb}{}^{x}=P_{yz}{}^{x}f_{c}{}^{y}f_{b}{}^{z}$$

if $p>1$.

Proof. If $m=n$, then the theorem is valid because of Lemma 1.1. Thus, we may only consider that $m\neq n$. In this case, we have $1-f_{x}f^{x}\neq 0$ on M.

Differentiating (1.34) covariantly and substituting (1.23), we find

$$(\nabla_{d}h_{ce}{}^{x})f^{e}+h_{ce}{}^{x}(f_{d}{}^{e}+h_{d}{}^{e}{}_{y}f^{y})=(\nabla_{d}\lambda^{x})f_{c}+\lambda^{x}(f_{dc}+h_{dc}{}^{y}f_{y}),$$

from which, taking the skew-symmetric part with respect to d and c,

(2.5) $$2h_{ce}{}^{x}f_{d}{}^{e}=(\nabla_{d}\lambda^{x})f_{c}-(\nabla_{c}\lambda^{x})f_{d}+2\lambda^{x}f_{dc}$$

with the help of (1.30), (2.2) and the fact that M has a flat normal connection.

Transvecting (2.5) with f^{c} and taking account of (1.34), we have

(2.6) $$\nabla_{d}\lambda^{x}=A^{x}f_{d},$$

A^{x} being defined by $A^{x}=f^{e}\nabla_{e}\lambda^{x}/1-f_{y}f^{y}$. Therefore, (2.5) becomes

(2.7) $$h_{ce}{}^{x}f_{d}{}^{e}=\lambda^{x}f_{dc}.$$

If we transvect (2.7) with $f_{b}{}^{d}$ and use (1.13), then we obtain

(2.8) $$h_{cb}{}^{x}=\lambda^{x}(g_{cb}-f_{c}{}^{y}f_{yb})+Q_{yz}{}^{x}f_{c}{}^{y}f_{b}{}^{z}$$

with the help of (1. 34) and (1. 35).
Transvecting (2. 8) with $f_w{}^b$ and making use of (1. 14) and (1. 35), we find

(2. 9) $$Q_{yz}{}^x f_c{}^y f^z = \lambda^x f_c{}^z f_z$$

because $f_x f^x$ is nonzero almost everywhere.

Transvecting this with $f_b{}^c$ and using (1. 15) and the fact that $f_a f^a$ is nonzero, we get

(2. 10) $$Q_{yz}{}^x f^y f^z = \lambda^x f_z f^z.$$

Also, transvecting (2. 9) with $f_w{}^c$ and using (1. 14), we find

(2. 11) $$Q_{yz}{}^x f^z = \lambda^x f_y$$

with the help of (2. 10), which implies

(2. 12) $$\lambda_x f_y - \lambda_y f_x = 0$$

because Q_{yzx} is symmetric for all indices.

Differentiation covariantly gives

$$(\nabla_d \lambda_x) f_y + \lambda_x \nabla_d f_y - (\nabla_d \lambda_y) f_x - \lambda_y \nabla_{dx} f_x = 0,$$

or, substituting (1. 24) and (2. 6) into this,

(2. 13) $$(A_x f_y - A_y f_x) f_d + \lambda_y f_{dx} - \lambda_x f_{dy} = 0$$

with the help of (1. 34). If we transvect (2. 13) with f^d and use (1. 17), then we find

$$A_x f_y - A_y f_x = 0$$

because of $f_a f^a \neq 0$. Consequently, (2. 13) reduces to

$$\lambda_y f_{dx} - \lambda_x f_{dy} = 0.$$

Transvecting this with $f_z{}^d$ and using (1. 14) and (2. 12), we get

$$\lambda_y g_{zx} - \lambda_x g_{zy} = 0.$$

This means $\lambda_y = 0$ if $p > 1$ and hence (2. 8) becomes

(2. 14) $$h_{cb}{}^x = P_{yz}{}^x f_c{}^y f_b{}^z$$

because of (1. 28) and (1. 36).
If $p=1$, that is, M is a hypersurface of S^{2m+1}, then M admits the so-called (f, g, u, v, λ)-structure (see [2], [9]). In this case, we have from (2. 9),

$$Q_{1^*1^*}{}^{1^*} f_c{}^{1^*} = \lambda^{1^*} f_c{}^{1^*}$$

because $(f^{1^*})^2$ is nonzero almost everywhere, which implies $Q_{1^*1^*}{}^{1^*} = \lambda^{1^*}$

because $f_{c1*}f^{c1*}=1-(f^{1*})^2$ does not vanish everywhere. Thus, (2. 8) is of the form

$$h_{cb}{}^{1*}=\lambda^{1*}g_{cb}, \tag{2. 15}$$

which shows that the hypersurface M is totally umbilical (cf. see [9]). This completes the proof of the lemma.

THEOREM 2. 2. *Let M be an n-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere S^{2m+1}. If the induced structure on M is normal, the function $f_x f^x$ does not vanish almost everywhere and the mean curvature vector is parallel in the normal bundle, then M is a $2m$-dimensional small sphere or an n-dimensional great sphere.*

Proof. If M is a hypersurface of S^{2m+1}, by Lemma 2. 1, M is a great or a small sphere. Thus, we only consider that the codimension $p>1$.

In this case, we have (2. 4) and $\lambda^x=0$, which obtained in the proof of Lemma 2. 1. And consequently $P_{yz}{}^x=Q_{yz}{}^x$ because of (1. 28) and (1. 36).

Transvecting (2. 4) with g^{cb} and using (1. 14) and (2. 11) with $\lambda^x=0$, we get

$$h^x=P^x, \tag{2. 16}$$

where we have put $P^x=g^{yz}P_{yz}{}^x$.

We have from (2. 4)

$$h_{ce}{}^x h_b{}^e{}_y=P_{uz}{}^x P_{wy}{}^z f_c{}^u f_b{}^w \tag{2. 17}$$

with the help of (2. 11) with $\lambda^x=0$.

Since the normal connection is flat, that is, $h_{ce}{}^x h_b{}^e{}_y-h_{be}{}^x h_c{}^e{}_y=0$, it follows that

$$(P_{uz}{}^x P_{wy}{}^z-P_{wz}{}^x P_{uy}{}^z)f_c{}^u f_b{}^w=0. \tag{2. 18}$$

Transvecting (2. 18) with $f_v{}^c f_w{}^b$ and using (1. 14) and (2. 11) with $\lambda^x=0$, we obtain

$$P_{vz}{}^x P_{wy}{}^z=P_{wz}{}^x P_{vy}{}^z. \tag{2. 19}$$

If we take account of (2. 4) and (2. 19), then (2. 17) becomes

$$h_{ce}{}^x h_b{}^e{}_y=P_{yz}{}^x h_{cb}{}^z, \tag{2. 20}$$

which and (2. 16) give

$$h_{cb}{}^x h^{cb}{}_x=h_x h^x. \tag{2. 21}$$

On the other hand, from the Ricci identity, we have

$$(g^{da}\nabla_d\nabla_a h_{cb}{}^x)h^{cb}{}_x-(\nabla_c\nabla_b h^x)h^{cb}{}_x=K_{ce}h_b{}^{ey}h^{cb}{}_y-K_{dcb}{}^a h_a{}^{dy}h^{cb}{}_y \tag{2. 22}$$

because of (2.2) and the fact that the normal connection is flat, where K_{ce} is the Ricci tensor given by

$$K_{ce}=(n-1)g_{ce}+h_x h_{ce}{}^x-h_c{}^a{}_x h_{ea}{}^x, \tag{2.23}$$

which is derived from (2.1). Hence, using the identity

$$\frac{1}{2}\Delta(h_{cb}{}^x h^{cb}{}_x)=(g^{da}\nabla_d\nabla_a h_{cb}{}^x)h^{cb}{}_x+\|\nabla_d h_{cb}{}^x\|^2,$$

we have

$$\frac{1}{2}\Delta(h_{cb}{}^x h^{cb}{}_x)=(\nabla_c\nabla_b h^x)h^{cb}{}_x+K_{ce}h_b{}^{ey}h^{cb}{}_y-K_{dcb}{}^a h_a{}^{dy}h^{cb}{}_y+\|\nabla_d h_{cb}{}^x\|^2,$$

where $\Delta=g^{cb}\nabla_c\nabla_b$. If we take account of (2.1), (2.3), (2.23) and the fact that the normal connection of M is flat, we have the following general formula:

$$\frac{1}{2}\Delta(h_{cb}{}^x h^{cb}{}_x)=(\nabla_c\nabla_b h^x)h^{cb}{}_x+nh_{cb}{}^x h^{cb}{}_x-h_x h^x+h^x h_{cex}h_b{}^{ey}h^{cb}{}_y$$
$$-(h_{cb}{}^x h^{cby})(h_{dax}h^{da}{}_y)+\|\nabla_d h_{cb}{}^x\|^2. \tag{2.24}$$

Since the mean curvature vector is parallel in the normal bundle, we see from (2.21) that $h_{cb}{}^x h^{cb}{}_x$ is constant. Thus, (2.24) reduces to

$$nh_{cb}{}^x h^{cb}{}_x-h_x h^x+h^x h_{cex}h_b{}^{ey}h^{cb}{}_y-(h_{cb}{}^x h^{cby})(h_{dax}h^{da}{}_y)+\|\nabla_d h_{cb}{}^x\|^2=0.$$

Substituting (2.20) and (2.21) into this and taking account of (2.16) and (2.19), we find $h^x=0$ and hence $h_{cb}{}^x=0$ because of (2.21). Therefore, by completeness, M is a great sphere. This completes the proof of the theorem.

From (2.23) we see that the scalar curvature K of M is given by

$$K=n(n-1) \tag{2.25}$$

because of (2.21). Thus combining (2.25), Remark B and Lemma 2.1, we conclude

REMARK C. Let M be an n-dimensional generic submanifold with $n\neq m$, $n\neq 2m$ and flat normal connection of an odd-dimensional unit sphere S^{2m+1}. If the induced structure on M is normal and $K\neq n(n-1)$, then we have $f^x=0$.

3. Generic submanifolds of an odd-dimensional sphere whose Sasakian structure vector is tangent to the submanifolds

In this section, we consider that M is an n-dimensional generic submanifold of S^{2m+1} with the Sasakian structure vector F^h given by (1.12) tangent to M, that is, $f^x=0$. Then (1.13)~(1.24) reduce to

$$f_c{}^e f_e{}^a = -\delta_c^a + f_c f^a + f_c{}^y f_y{}^a, \tag{3.1}$$

$$f_c{}^e f_e{}^x = 0, \tag{3.2}$$

$$f_e{}^x f^e = 0, \tag{3.3}$$

$$f_e{}^a f^e = 0, \tag{3.4}$$

$$f_x{}^e f_e{}^y = \delta_x^y, \tag{3.5}$$

$$f_e f^e = 1, \tag{3.6}$$

$$\nabla_c f_b{}^a = -g_{cb} f^a + \delta_c^a f_b + h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x, \tag{3.7}$$

$$\nabla_c f_b{}^x = h_{ce}{}^x f_b{}^e, \quad \nabla_c f_x{}^a = -h_c{}^e{}_x f_e{}^a, \tag{3.8}$$

$$h_c{}^e f_e{}^y = h_{ce}{}^y f_x{}^e, \tag{3.9}$$

$$\nabla_c f^a = f_c{}^a, \tag{3.10}$$

$$f_c{}^x + h_{ce}{}^x f^e = 0. \tag{3.11}$$

From (3.1) and (3.2), we can easily find that

$$f_c{}^e f_e{}^a f_a{}^b + f_c{}^b = 0,$$

and consequently M admits the so-called f-structure satisfying $f^3+f=0$.

We first prove

LEMMA 3.1. *Let M be an n-dimensional generic submanifold of an odd-dimensional unit sphere S^{2m+1} with flat normal connection. If the structure induced on M with the Sasakian structure vector tangent to M is normal, then we have*

$$h_{ce}{}^x h_b{}^e{}_y = P_{yz}{}^x h_{cb}{}^z + \delta_y^x g_{cb}, \tag{3.12}$$

where we have put

$$P_{yz}{}^x = h_{cb}{}^x f_y{}^c f_z{}^b. \tag{3.13}$$

Proof. Transvecting (1.30) with $f_y{}^c$ and using (3.2), we find

$$h_{ce}{}^x f_y{}^c f_b{}^e = 0,$$

from which, transvecting $f_a{}^b$ and using (3.1),

$$h_{ce}{}^x f_y{}^e = P_{yw}{}^x f_c{}^w - \delta_y^x f_c \tag{3.14}$$

because of (3.5), (3.11) and (3.13).

Putting $P_{yzx} = P_{yz}{}^w g_{wx}$, we can easily find that P_{yzx} is symmetric for all indices because of (3.9) and (3.13).

If we transvect (3.14) with $h_b{}^c{}_z$ and make use of (3.11) and (3.14), then we have

$$h_b{}^c{}_z h_{ce}{}^x f_y{}^e = P_{yw}{}^x P_{zv}{}^w f_b{}^v - P_{yz}{}^x f_b + \delta_y^x f_{bz},$$

from which, using (3.11), (3.14) and the fact that the normal connection of M is flat,

$$P_{yz}{}^{w}P_{wv}{}^{x}f_{b}{}^{v}+g_{yz}f_{b}{}^{x}=P_{yw}{}^{x}P_{zv}{}^{w}f_{b}{}^{v}+\delta_{y}{}^{x}f_{bz},$$

or, transvecting $f_{u}{}^{b}$ and taking account of (3.5),

$$P_{yz}{}^{w}P_{wu}{}^{x}+g_{yz}\delta_{u}{}^{x}=P_{yw}{}^{x}P_{zu}{}^{w}+\delta_{y}{}^{x}g_{uz} \tag{3.15}$$

because P_{yzx} is symmetric for all indices. Thus, it follows that

$$P_{zxw}P_{y}{}^{wx}=P_{x}P_{zy}{}^{x}+(p-1)g_{zy}. \tag{3.16}$$

Differentiating (3.14) covariantly along M and substituting (3.8) and (3.10), we get

$$(\nabla_{d}h_{ce}{}^{x})f_{y}{}^{e}+h_{c}{}^{ex}h_{day}f_{e}{}^{a}=(\nabla_{d}P_{yz}{}^{x})f_{c}{}^{z}+P_{yz}{}^{x}h_{de}{}^{z}f_{c}{}^{e}-\delta_{y}{}^{x}f_{dc},$$

from which, taking the skew-symmetric part with respect to d and c and using (2.2),

$$-2h_{c}{}^{ex}h_{eay}f_{d}{}^{a}=(\nabla_{d}P_{yz}{}^{x})f_{c}{}^{z}-(\nabla_{c}P_{yz}{}^{x})f_{d}{}^{z}-2P_{yz}{}^{x}h_{ce}{}^{z}f_{d}{}^{e}-2\delta_{y}{}^{x}f_{dc} \tag{3.17}$$

with the help of (1.30) and (2.3) with $K_{dcy}{}^{x}=0$. Transvecting (3.17) with $f_{w}{}^{d}$ and using (3.2) and (3.5), we get

$$\nabla_{c}P_{yw}{}^{x}=(f_{w}{}^{e}\nabla_{e}P_{yz}{}^{x})f_{c}{}^{z}, \tag{3.18}$$

which and $P_{yz}{}^{x}=P_{zy}{}^{x}$ imply

$$(\nabla_{c}P_{yz}{}^{x})f_{b}{}^{z}=(f_{y}{}^{e}\nabla_{e}P_{wz}{}^{x})f_{c}{}^{z}f_{b}{}^{w}.$$

Therefore (3.17) becomes

$$h_{c}{}^{ex}h_{eay}f_{d}{}^{a}=P_{yz}{}^{x}h_{ce}{}^{z}f_{d}{}^{e}+\delta_{y}{}^{x}f_{dc}.$$

If we apply $f_{b}{}^{d}$ to this and use (3.1), we obtain

$$-h_{c}{}^{ex}h_{bey}+P_{yz}{}^{w}P_{vw}{}^{x}f_{c}{}^{v}f_{b}{}^{z}+f_{c}{}^{x}f_{by}=-P_{yz}{}^{x}h_{cb}{}^{z}+P_{yw}{}^{x}P_{vz}{}^{w}f_{b}{}^{z}f_{c}{}^{v}$$
$$-\delta_{y}{}^{x}g_{cb}+\delta_{y}{}^{x}f_{c}{}^{z}f_{zb}$$

with the help of (3.11), or, take account of (3.15), we find (3.12). Hence, Lemma 3.1 is proved.

LEMMA 3.2. *Under the same assumptions as those stated in Lemma* 3.1, *we have*

$$\nabla_{c}h^{x}=\nabla_{c}P^{x}, \tag{3.19}$$

where $P^{x}=g^{yz}P_{yz}{}^{x}$.

Proof. Differentiating (1.30) covariantly and substituting (3.7), we find

$$(\nabla_{d}h_{ce}{}^{x})f_{b}{}^{e}+h_{ce}{}^{x}(-g_{db}f^{e}+\delta_{d}{}^{e}f_{b}+h_{db}{}^{y}f_{y}{}^{e}-h_{d}{}^{e}{}_{y}f_{b}{}^{y})+(\nabla_{d}h_{be}{}^{x})f_{c}{}^{e}$$
$$+h_{be}{}^{x}(-g_{dc}f^{e}+\delta_{d}{}^{e}f_{c}+h_{dc}{}^{y}f_{y}{}^{e}-h_{d}{}^{e}{}_{y}f_{c}{}^{y}),$$

from which, taking account of (3.11), (3.12) and (3.14),

$$(\nabla_d h_{ce}{}^x) f_b{}^e + (\nabla_d h_{be}{}^x) f_c{}^e = 0,$$

or, taking the skew-symmetric part with respect to the indices d and c, and using (2.2),

$$(\nabla_d h_{be}{}^x) f_c{}^e - (\nabla_c h_{be}{}^x) f_d{}^e = 0.$$

The last two equations give $(\nabla_d h_{ce}{}^x) f_b{}^e = 0$ by means of (2.2). Transvection $f_a{}^b$ yields

$$\nabla_d h_{ca}{}^x = (\nabla_d h_{ce}{}^x) f_y{}^e f_a{}^y + (\nabla_d h_{ce}{}^x) f^e f_a$$

with the help of (3.1), which implies

$$\nabla_d h^x = (\nabla_d h_{ce}{}^x) f_y{}^e f^{cy} + (\nabla_d h_{ce}{}^x) f^e f^c. \tag{3.20}$$

But, we see from (3.3) and (3.11) that $h_{ce}{}^x f^c f^e = 0$. Differentiating this covariantly and making use of (3.4) and (3.10), we get $(\nabla_d h_{ce}{}^x) f^c f^e = 0$. Consequently (3.20) becomes

$$\nabla_d h^x = (\nabla_d h_{ce}{}^x) f_y{}^e f^{cy}. \tag{3.21}$$

On the other hand, we have from (3.13)

$$P^x = h_{ce}{}^x f_y{}^e f^{cy}.$$

If we differentiate this covariantly and take account of (3.8), we find

$$\nabla_d P^x = (\nabla_d h_{ce}{}^x) f_y{}^e f^{cy} + 2 h_c{}^{ex} h_{day} f_e{}^a f^{cy},$$

which means

$$\nabla_d P^x = (\nabla_d h_{ce}{}^x) f_y{}^e f^{cy}$$

with the help of (3.5) and (3.11). Thus, this together with (3.21) gives (3.19). Thus, Lemma 3.2 is completely proved.

If we substitute (3.12) into (2.24) and make use of (3.16), we find

$$\frac{1}{2}\Delta (h_{cb}{}^x h^{cb}{}_x) = (\nabla_c \nabla_b h^x) h^{cb}{}_x + \|\nabla_d h_{cb}{}^x\|^2. \tag{3.22}$$

Assuming that the mean curvature vector is parallel in the normal bundle, we see from (3.19) that $\nabla_c P^x = 0$. And hence

$$h_{cb}{}^x h^{cb}{}_x = h_x h^x + n(2m+1-n), \tag{3.23}$$

which is derived from (3.12), is a constant. Consequently (3.22) implies that the second fundamental tensors are parallel. Moreover, M is of essential codimension $2m+1-n$ and does not admit umbilical sections because of (3.14). Combining these facts with Theorem A in §0, we obtain

THEOREM 3. 3. *Let M be an n-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere* S^{2m+1} *and let the Sasakian structure vector defined on* S^{2m+1} *be tangent to M. If the structure induced on M is normal and if the mean curvature vector of M is parallel in the normal bundle, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\ldots\times S^{p_N}(r_N),$$

where $p_1, \ldots, p_N$ *are odd numbers* $\geqq 1$, $r_1{}^2+\cdots+r_N{}^2=1$, $N=2m-n+2$.

Combining Remark *C* with Theorem 3. 3, we have

COROLLARY 3. 4. *Let M be an n-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere* S^{2m+1} $(n\neq m, n\neq 2m)$. *If the structure induced on M is normal, the scalar curvature K of M satisfies* $K\neq n(n-1)$ *and if the mean curvature vector of M is parallel in the normal bundle, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),$$

where $p_1, \ldots, p_N$ *are odd numbers* $\geqq 1$, $r_1{}^2+\cdots+r_N{}^2=1$, $N=2m-n+2\,(N\neq 2, N\neq n+2)$.

Moreover, if the submanifold is minimal, then we see from (3. 22) and (3. 23), that the second fundamental tensors are parallel. Therefore, making use of Theorem *B* and Remark *A* in § 0, we have

THEOREM 3. 5. *Let M be an n-dimensional complete minimal generic submanifold with flat normal connection of an odd-dimensional unit sphere* S^{2m+1} *and let the Sasakian structure vector defined on* S^{2m+1} *be tangent to M. If the structure induced on M is normal, then M is a great sphere of* S^{2m+1} *or a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),$$

$p_1, \ldots, p_N$ *are odd numbers* $\geqq 1$, $p_1+\cdots+p_N=n$, $1<N\leqq 2m-n+2$ *with essential codimension* $N-1$, *where* $r_t=\sqrt{p_t/n}\,(t=1, \ldots, N)$.

4. Partially integrable generic submanifolds of an odd-dimensional sphere

In this section, first of all we prove

LEMMA 4. 1. *Let M be an n-dimensional generic submanifold with flat normal connection of* S^{2m+1}. *If the scalar curvature K of M satisfies* $K\neq n(n-1)$, *then the function* $1-f_xf^x$ *does not vanish almost everywhere on M.*

Proof. As a matter of convenience, we denote by $M_0=\{p\in M|\,(1-f_xf^x)$

$(p)=0\}$.

Suppose that M_0 is nonempty. Let W be a connected component of M_0. Then we have $f_c=0$ on W because of (1.18). Thus (1.23) implies

$$h_{cb}{}^x f_x=0 \text{ and } f_{cb}=0 \tag{4.1}$$

on W because $h_{cb}{}^x f_x$ is symmetric and f_{cb} is skew-symmetric with respect to the indices c and b. Therefore, (1.13) and (1.19) reduce respectively to

$$f_b{}^x f_x{}^a=\delta_b{}^a, \tag{4.2}$$

$$h_{cb}{}^x f_x{}^a-h_c{}^a{}_x f_b{}^x=0 \tag{4.3}$$

on W. As is shown in Lemma 1.1, we have

$$h_{cb}{}^x=P_{yz}{}^x f_c{}^y f_b{}^z. \tag{4.4}$$

Putting $P_{yzx}=P_{yz}{}^w g_{wx}$, P_{yzx} is symmetric for any index because of (1.22). Hence it follows that

$$h_{cb}{}^x h^{cb}{}_x=P_{xyz}P^{xyz} \tag{4.5}$$

on W with the help of (1.14), (1.16) and (4.1).

If we transvect (4.4) with g^{cb} and use (1.14) and (1.16), we have $h^x=P^x$.

Since the normal connection is flat, we have from (2.3)

$$h_{de}{}^x h_c{}^e{}_y-h_{ce}{}^x h_d{}^e{}_y=0. \tag{4.6}$$

Transvecting this with $f_z{}^d f_u{}^c$ and using (4.1) and (4.4), we get

$$P_{wyz}P_{ux}{}^w=P_{wxz}P_{uy}{}^w$$

on W, from which, we obtain

$$P_{xyz}P^{xyz}=h_x h^x. \tag{4.7}$$

On the other hand, the Ricci tensor K_{cb} of M is given by

$$K_{cb}=(n-1)g_{cb}+h_x h_{cb}{}^x-h_{ce}{}^x h_b{}^e{}_x.$$

Therefore, the scalar curvature K of M is equal to $n(n-1)$ on W because of (4.5) and (4.7). Consequently, the set W is empty. It contradicts that W is nonempty. This completes the proof of our lemma.

We now define a tensor field $N_{cb}{}^x$ of type (0.2) for fixed index x by

$$N_{cb}{}^x=f_c{}^e\nabla_e f_b{}^x-f_b{}^e\nabla_e f_c{}^x-(\nabla_c f_b{}^e-\nabla_b f_c{}^e)f_e{}^x-(\nabla_c f_b-\nabla_b f_c)f^x. \tag{4.8}$$

If the tensor $N_{cb}{}^x$ vanishes identically, the aggregate $(f_c{}^a, g_{cb}, f_c{}^x, f_c, f^x)$ is partially integrable (see [3]).

From now on we assume that the induced structure satisfying (1.13)～

(1.18) on M is partially integrable. Then we have

$$(4.9)\qquad (h_{cey}f^{ex})f_b^{\ y}=(h_{bey}f^{ex})f_c^{\ y}+f_c^{\ x}f_b-f_b^{\ x}f_c$$

because of (1.19), (1.20), (1.23) and (4.8). Transvecting (4.9) with $f_z^{\ b}$ and using (1.14) and (1.17), we find

$$(4.10)\qquad h_{cez}f^{ex}-(h_{cey}f^{y})f^{ex}f_z=P_{yz}^{\ \ x}f_c^{\ y}-\delta_z^{\ x}f_c+f_zf^xf_c,$$

from which, transvecting f^z to this and combining these expressions,

$$(1-f_yf^y)h_{cez}f^{ex}=-(1-f_yf^y)\delta_z^{\ x}f_c+\{(1-f_yf^y)P_{zw}^{\ \ x}+f_zP_{wu}^{\ \ x}f^u\}f_c^{\ w},$$

or, equivalently

$$(1-f_yf^y)h_{cez}f_x^{\ e}=-(1-f_yf^y)g_{xz}f_c+\{(1-f_yf^y)P_{zwx}+f_zP_{wux}f^u\}f_c^{\ w}.$$

Taking the skew-symmetric part with respect to the indices x and z and using (1.22), we obtain

$$(4.12)\qquad (P_{ywx}f_zf^w-P_{ywz}f_xf^w)f_c^{\ y}=0.$$

LEMMA 4.2 ([3]). *Let M be a generic submanifold with flat normal connection of S^{2m+1}. If the induced structure $(f_b^{\ a}, g_{cb}, f_c^{\ x}, f_c, f_x)$ on M is partially integrable and $K\neq n(n-1)$ holds, then we have*

$$(4.13)\qquad h_{ce}^{\ \ x}f_y^{\ e}=-\delta_y^{\ x}f_c+R_{yz}^{\ \ x}f_c^{\ z},$$

$$(4.14)\qquad R_{wz}^{\ \ x}R_{uy}^{\ \ w}-R_{wyz}R_u^{\ xw}=\{(1-f_wf^w)\delta_u^{\ x}+f_uf^x\}g_{yz}-\{(1-f_wf^w)g_{yu}+f_yf_u\}\delta_z^{\ x},$$

where R_{yzx} is symmetric for any index and given by

$$(4.15)\qquad R_{yzx}=P_{yzx}+P_{yux}f^uf_z/(1-f_wf^w).$$

Proof. Since $K\neq n(n-1)$, we have seen in Lemma 4.1 that the function $1-f_wf^w$ does not vanish almost everywhere on M. And hence (4.11) implies (4.13) and (4.15).

Transvecting (4.12) with $f_v^{\ c}$ and $f_a^{\ c}$ respectively and combining these expressions, we find

$$(4.16)\qquad (f_zP_{ywx}-f_xP_{ywz})f^w=0$$

with the aid of (1.14), (1.15) and Lemma 4.1.

Using this, we find that R_{yzx} is symmetric for all indices.

Since the normal connection is flat, that is, $h_{ce}^{\ \ x}h_b^{\ e}{}_y-h_{be}^{\ \ x}h_c^{\ e}{}_y=0$, by transvecting this with $f_z^{\ c}$, we find

$$(4.17)\qquad (R_{wz}^{\ \ x}R_{vy}^{\ \ w}-R_{wyz}R_v^{\ xw})f_d^{\ v}=\delta_z^{\ x}(h_{dey}f^e)-g_{yz}(h_{de}^{\ \ x}f^e)$$

with the aid of (4.13). Transvecting (4.17) with $f_u^{\ d}$ and $f_u^{\ d}f^u$ respectively and combining these results, we obtain (4.14) because of (1.14) and

(4. 13). Thus, Lemma 4. 2 is completely proved.

5. Partially integrable Ricci parallel generic submanifolds of an odd-dimensional sphere tangent to the Sasakian structure vector field

In this section. we assume that the structure $(f_c{}^a, g_{cb}, f_c{}^x, f_c, f_x)$ induced on an n-dimensional generic submanifold M of S^{2m+1} is partially integrable, the connection in the normal bundle of M is flat and its Sasakian structure vector field F^h is tangent to M, that is, $f_x=0$. And hence (3. 1)~(3. 11) are valid. Also, (4. 13) and (4. 14) reduce respectively to

$$h_{ce}{}^x f_y{}^e = -\delta_y{}^x f_c + P_{yz}{}^x f_c{}^z \tag{5.1}$$

and

$$P_{wz}{}^x P_{uy}{}^w - P_{wyz} P_u{}^{xw} = \delta_u{}^x g_{yz} - g_{yu}\delta_z{}^x \tag{5.2}$$

provided that the scalar curvature K satisfies $K \neq n(n-1)$.
It follows from (5. 2) immediately that

$$P_{wxz} P_y{}^{xw} = P_{zyx} P^x + (2m-n) g_{yz} \tag{5.3}$$

since P_{yzx} is symmetric for all indices.

On the other hand, the Ricci tensor K_{cb} was obtained in §2 of the form

$$K_{cb} = (n-1) g_{cb} + h^x h_{cbx} - h_{ce}{}^x h_b{}^e{}_x. \tag{5.4}$$

If we transvect (5. 4) with f^b and $f_y{}^b$ respectively and make use of (3. 11) and (5. 1), then we have

$$K_{ce} f^e = 2(n-m-1) f_c + (P_x - h_x) f_c{}^x \tag{5.5}$$

and

$$K_{ce} f_y{}^e = 2(n-m-1) f_{yc} + (P_y - h_y) f_c + P_{zyx}(h^x - P^x) f_c{}^z \tag{5.6}$$

respectively with the aid of (5. 3).

Now, suppose that the Ricci tensor is parallel along M, that is, $\nabla_d K_{cb}=0$.

Differentiating (5. 5) covariantly along M and substituting (3. 8) and (3. 10), we find

$$K_{ce} f_b{}^e = 2(n-m-1) f_{bc} + \{\nabla_b (P_x - h_x)\} f_c{}^x + (P_x - h_x) h_{be}{}^x f_c{}^e. \tag{5.7}$$

If we transvect (5. 6) and (5. 7) with $f_b{}^c$ and $f_y{}^c$ respectively and compare these expressions, then we obtain

$$\nabla_b P_x = \nabla_b h_x. \tag{5.8}$$

Thus, (5. 7) becomes

(5. 9) $$K_{ce}f_b{}^e=2(n-m-1)f_{bc}+(P_x-h_x)h_{be}{}^xf_c{}^e.$$

If we differentiate (5. 6) covariantly along M and take account of (3. 8) and (3. 10), then we have

(5. 10) $$K_c{}^eh_{bay}f_e{}^a=2(n-m-1)h_{bey}f_c{}^e+(P_y-h_y)f_{bc}+(\nabla_bP_{zyx})(h^x-P^x)f_c{}^z+P_{zyx}(h^x-P^x)h_{be}{}^zf_c{}^e$$

with the aid of $\nabla_dK_{cb}=0$ and (5. 8).
Transvection with $f_w{}^c$ gives

(5. 11) $$(\nabla_bP_{zyx})(h^x-P^x)=0$$

because of (3. 27) and (5. 9). And consequently, (5. 10) reduces to

(5. 12) $$K_c{}^eh_{bay}f_e{}^a=2(n-m-1)h_{bey}f_c{}^e+(P_y-h_y)f_{bc}+P_{zyx}(h^x-P^x)h_{be}{}^zf_c{}^e.$$

Now we prove the following lemma.

LEMMA 5. 1. *Let M be an n-dimensional partially integrable Ricci parallel generic submanifold of S^{2m+1} with flat normal connection, on which the Sasakian structure vector is tangent to M. If the scalar curvature K satisfies $K\neq n(n-1)$, then we have*

(5. 13) $$h_{cb}{}^xh^{cb}{}_x=P_xh^x+n(2m+1-n).$$

Proof. If we substitute (5. 3) into (5. 9), then we find

(5. 14) $$h_{ca}{}^xh_e{}^a{}_xf_b{}^e-h_x(h_{ce}{}^xf_b{}^e+h_{be}{}^xf_c{}^e)-(2m+1-n)f_{bc}+P_xh_{be}{}^xf_c{}^e=0.$$

Transvecting (5. 14) with $f_d{}^b$ and using (3. 1), (3. 11) and (5. 1), we get

$$\begin{aligned}&h_{ca}{}^xh_d{}^a{}_x+(P_x-h_x)(f_c{}^xf_d+f_d{}^xf_c)-h^x(h_{cdx}-P_{xyz}f_c{}^zf_d{}^y)\\&+(h^x-P^x)h_{bex}f_d{}^bf_c{}^e-(2m+1-n)g_{cd}+(2m-n)f_d{}^xf_{xc}\\&-P_{zxy}P_w{}^{zx}f_c{}^wf_d{}^y=0,\end{aligned}$$

or using (5. 3),

$$\begin{aligned}&h_{ca}{}^xh_d{}^a{}_x+(P_x-h_x)(f_c{}^xf_d+f_d{}^xf_c)-(P_x-h_x)P_{yz}{}^xf_c{}^zf_d{}^y\\&-h^xh_{cdx}+(h^x-P^x)h_{be}{}^xf_d{}^bf_c{}^e-(2m+1-n)g_{dc}=0.\end{aligned}$$

If we transvect this with g^{cd} and utilize (3. 1), (3. 3) and (3. 5), then we find

$$h_{ca}{}^xh^{ca}{}_x-h_xP^x-n(2m+1-n)=0.$$

Thus, Lemma 5. 1 is proved.

LEMMA 5. 2. *Under the same assumptions as those stated in Lemma 5. 1, we have*

(5. 15) $$h_{ce}{}^xf_b{}^e+h_{be}{}^xf_c{}^e=0,$$

that is, M admits the normality.

Proof. We consider the following identity:

$$(5.16)\quad \nabla^b(f_x{}^c\nabla_c f_b{}^x)=\frac{1}{2}\|\nabla_c f_b{}^x+\nabla_b f_c{}^x\|^2-\|\nabla_c f_b{}^x\|^2+f_x{}^c\nabla^b\nabla_c f_b{}^x$$
$$=\frac{1}{2}\|h_{ce}{}^x f_b{}^e+h_{be}{}^x f_c{}^e\|^2-\|h_{be}{}^x f_b{}^e\|^2+f_x{}^c\nabla^b\nabla_c f_b{}^x$$

because of (3.8).

Now, we compute the square of norm of $\nabla_c f_b{}^x$ by using (3.1) and (3.5):

$$(5.17)\quad \|\nabla_c f_b{}^x\|^2=-h_{ce}{}^x f_b{}^e h^{ca}{}_x f_a{}^b=h_{ca}{}^x h^{ca}{}_x-P_xP^x-(2m+1-n)(2m-n)$$

with the aid of (5.1) and (5.3).

The Ricci identity for $f_b{}^x$ implies

$$(5.18)\quad \nabla_d\nabla_c f_b{}^x-\nabla_c\nabla_d f_b{}^x=-K_{dcb}{}^e f_e{}^x$$

because the normal connection of M is flat. Transvection g^{db} yields

$$\nabla^b\nabla_c f_b{}^x=K_{ce}f^{ex}=2(n-m-1)f_c{}^x+(P^x-h^x)f_c+P_{zy}{}^x(h^y-P^y)f_c{}^z$$

because of (5.6). If we transvect this with $f_x{}^c$ and use (3.4) and (3.5), then we have

$$(5.19)\quad f_x{}^c\nabla^b\nabla_c f_b{}^x=2(2m+1-n)(n-m-1)+P_x(h^x-P^x).$$

Since the left side of (5.16) is zero because of (3.2), (3.8) and (5.1), if we substitute (5.17) and (5.19) into (5.16), then we have

$$\frac{1}{2}\|h_{ce}{}^x f_b{}^e+h_{be}{}^x f_c{}^e\|^2-h_{ca}{}^x h^{ca}{}_x+h_xP^x+n(2m+1-n)=0.$$

Combining this fact and Lemma 5.1, we obtain

$$h_{ce}{}^x f_b{}^e+h_{be}{}^x f_c{}^e=0,$$

that is, the induced partially integrable structure on M is normal. Therefore, Lemma 5.2 is proved.

Thus, regarding to Theorem 3.3 and Theorem 3.5, we have

THEOREM 5.3. *Let M be an n-dimensional partially integrable Ricci parallel complete generic submanifold with flat normal connection of S^{2m+1} tangent to the Sasakian structure vector field. If the scalar curvature K satisfies $K\neq n(n-1)$ and the mean curvature vector of M is parallel in the normal bundle, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),$$

where $p_1,\dots p_N$ are odd numbers $\geqq 1$, $r_1{}^2+\cdots+r_N{}^2=1$, $N=2m-n+2$.

THEOREM 5.4. *Let M be an n-dimensional complete minimal Ricci parallel generic submanifold with flat normal connection of* S^{2m+1} *tangent to the Sasakian structure vector field. If the scalar curvature* $K\neq n(n-1)$ *and the structure induced on M is partially integrable, then M is a great sphere of* S^{2m+1} *or a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),$$

$p_1, \dots, p_N$ *are odd numbers* $\geqq 1$, $p_1+\cdots+p_N=n$, $1<N\leqq 2m-n+2$ *with essential codimension* $N-1$, *where* $r_t=\sqrt{p_t/n}\,(t=1, \dots, N)$.

COROLLARY 5.5. *In Theorem* 5.3, *if* $n\neq m$ *and* $n\neq 2m$, *then* $N\neq 2$ *and* $N\neq n+2$.

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