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AN APPLICATION OF THE FRACTIONAL CALCULUS III

SHIGEYOSHI OWA

I. Introduction

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

that are analytic and univalent in the unit disk $\mathcal{U}=\{z:|z|<1\}$. Then a function $f(z)\in\mathcal{S}$ is said to be starlike of order α $(0\leqq\alpha<1)$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (z\in\mathcal{U}) \tag{1.2}$$

and a function $f(z)\in\mathcal{S}$ is said to be convex of order α $(0\leqq\alpha<1)$ if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \qquad (z\in\mathcal{U}). \tag{1.3}$$

And we denote by $\mathcal{S}^*(\alpha)$ the class of all starlike functions of order α and by $\mathcal{K}(\alpha)$ the class of all convex functions of order α.

The class $\mathcal{S}^*(\alpha)$ was first studied by Robertson [8] and latter Merkes, Robertson and Scott [5], Schild [11], Bogowski, Jablonski and Stankiewicz [1] and Jack [3]. In particular, the class $\mathcal{S}^*(1/2)$ was studied by Schild [12] and MacGregor [4]. Further the subclass $\mathcal{T}^*(\alpha)$ of $\mathcal{S}^*(\alpha)$ and the subclass $\mathcal{C}(\alpha)$ of $\mathcal{K}(\alpha)$ consisting of analytic and univalent functions with negative coefficients were studied by Silverman [13].

We say that a function $f(z)$ defined by (1.1) is in the class $\mathcal{V}(\theta_n)$ if $f(z)\in\mathcal{S}$ and $\arg(a_n)=\theta_n$ for all $n\geqq2$. If, further, there exists a real number β such that

$$\theta_n+(n-1)\beta\equiv\pi\,(\mathrm{mod}\,2\pi) \tag{1.4}$$

then $f(z)$ is said to be in the class $\mathcal{V}(\theta_n;\beta)$. The union of $\mathcal{V}(\theta_n;\beta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers β is denoted by $\mathcal{V}$.

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Let $\mathcal{T}^*(\alpha)$ denote the subclass of $\mathcal{T}$ consisting of starlike functions of order α. And let $\mathcal{C}(\alpha)$ denote the subclass of $\mathcal{T}$ consisting of convex functions of order α.

For these classes, Silverman [14] gave many interesting results and three open problems.

Now, several essentially equivalent definitions of the fractional calculus have been given in the literature (cf., e. g., [2], [6], [9], [10] and [15]). We find it convenient to restrict ourselves to the following definitions of the fractional calculus used recently by Owa [7].

DEFINITION 1. The fractional integral of order λ is defined by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\lambda}}, \tag{1.5}$$

where $\lambda>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 2. The fractional derivative of order λ is defined by

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{\lambda}} \tag{1.6}$$

where $0<\lambda<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z), \tag{1.7}$$

where $0<\lambda<1$ and $n\in\mathcal{N}\cup\{0\}$.

2. The classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$

Recently Silverman [14] showed the following lemmas for functions in $\mathcal{T}^*(\alpha)$.

LEMMA 1. *If the function $f(z)$ defined by (1.1) is in the class $\mathcal{T}^*(\alpha)$, then*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leqq 1-\alpha. \tag{2.1}$$

LEMMA 2. *If the function $f(z)$ defined by (1.1) is in the class $\mathcal{T}^*(\alpha)$, then*

(2.2) $$|z|-\frac{1-\alpha}{2-\alpha}|z|^2\leqq|f(z)|\leqq|z|+\frac{1-\alpha}{2-\alpha}|z|^2$$

and

(2.3) $$1-\frac{2(1-\alpha)}{2-\alpha}|z|\leqq|f'(z)|\leqq 1+\frac{2(1-\alpha)}{2-\alpha}|z|$$

for $z\in\mathcal{U}$. *Equality in all cases occurs for*

(2.4) $$f(z)=z+\frac{1-\alpha}{2-\alpha}e^{i\theta_2}z^2$$

at $z=\pm|z|e^{-i\theta_2}$.

THEOREM 1. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{Q}(\alpha)$, *then*

(2.5) $$\sum_{n=2}^{\infty} n(n-\alpha)|a_n|\leqq 1-\alpha.$$

Proof. We employ the same technique as used by Silverman [14]. We assume that

(2.6) $$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}=Re\left\{\frac{1+\sum_{n=2}^{\infty}n^2a_nz^{n-1}}{1+\sum_{n=2}^{\infty}na_nz^{n-1}}\right\}>\alpha$$

for $z\in\mathcal{U}$. For $f(z)\in\mathcal{Q}(\theta_n;\beta)$ we set $z=re^{i\beta}$ in (2.6) and let $r\to 1^-$. Upon clearing the denominator in (2.6) we obtain

(2.7) $$1-\sum_{n=2}^{\infty}n^2|a_n|\geqq\alpha\left(1-\sum_{n=2}^{\infty}n|a_n|\right)$$

which shows (2.5). This completes the proof of the theorem.

THEOREM 2. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{Q}(\alpha)$, *then*

(2.8) $$|z|-\frac{1-\alpha}{2(2-\alpha)}|z|^2\leqq|f(z)|\leqq|z|+\frac{1-\alpha}{2(2-\alpha)}|z|^2$$

and

(2.9) $$1-\frac{1-\alpha}{2-\alpha}|z|\leqq|f'(z)|\leqq 1+\frac{1-\alpha}{2-\alpha}|z|$$

for $z\in\mathcal{U}$. *Equality in all cases occurs for*

(2.10) $$f(z)=z+\frac{1-\alpha}{2(2-\alpha)}e^{i\theta_2}z^2$$

at $z=\pm|z|e^{-i\theta_2}$.

Proof. We use a method of Silverman [13]. In view of Theorem 1, we can see that

$$(2.11)\qquad \sum_{n=2}^{\infty}|a_n|\leqq\frac{1-\alpha}{2(2-\alpha)}.$$

Hence, by using (2.11), we have

$$(2.12)\qquad |f(z)|\leqq|z|+\sum_{n=2}^{\infty}|a_n||z|^n\leqq|z|+|z|^2\sum_{n=2}^{\infty}|a_n|\leqq|z|+\frac{1-\alpha}{2(2-\alpha)}|z|^2$$

which gives the right hand side of (2.8).

Similary, we get

$$(2.13)\qquad |f(z)|\geqq|z|-\sum_{n=2}^{\infty}|a_n||z|^n\geqq|z|-|z|^2\sum_{n=2}^{\infty}|a_n|\geqq|z|-\frac{1-\alpha}{2(2-\alpha)}|z|^2$$

which gives the left hand side of (2.8).

Further, with the aid of Theorem 1, we have

$$(2.14)\qquad \sum_{n=2}^{\infty}n|a_n|\leqq\frac{1-\alpha}{2-\alpha}.$$

Consequently, by means of (2.14),

$$(2.15)\qquad |f'(z)|\leqq1+\sum_{n=2}^{\infty}n|a_n||z|^{n-1}\leqq1+|z|\sum_{n=2}^{\infty}n|a_n|\leqq1+\frac{1-\alpha}{2-\alpha}|z|.$$

On the other hand,

$$(2.16)\qquad |f'(z)|\geqq1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}\geqq1-|z|\sum_{n=2}^{\infty}n|a_n|\geqq1-\frac{1-\alpha}{2-\alpha}|z|.$$

Thus we have the theorem.

COROLLARY 1. *Under the hypotheses of* Theorem 2, $f(z)$ *is included in a disk with its center at the origin and radius* $(5-3\alpha)/2(2-\alpha)$, *and* $f'(z)$ *is included in a disk with its center at the origin and radius* $(3-2\alpha)/(2-\alpha)$.

3. Application of the fractional calculus

For $f(z)$ defined by (1.1), we put

$$(3.1)\qquad \begin{aligned}F(z)&=\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}f(z)\\&=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}a_nz^n\end{aligned}\qquad (\lambda>0)$$

and

$$(3.2)\qquad G(z)=\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z) = z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}a_n z^n \qquad (0<\lambda<1).$$

Now, we need the following lemma for functions in $\mathcal{P}$ by Silverman [14].

LEMMA 3. *If the function $f(z)$ defined by* (1.1) *is in the class* $\mathcal{P}$, *then*

$$(3.3)\qquad \sum_{n=2}^{\infty} n|a_n|\leqq 1.$$

THEOREM 3. *If the function $f(z)$ defined by* (1.1) *is in the class* $\mathcal{P}$, *then*

$$(3.4)\qquad \frac{|z|^{1+\lambda}}{2\Gamma(2+\lambda)}(2-|z|)\leqq |D_z^{-\lambda}f(z)|\leqq\frac{|z|^{1+\lambda}}{2\Gamma(2+\lambda)}(2+|z|)$$

for $\lambda>0$ *and* $z\in\mathcal{U}$, *and*

$$(3.5)\qquad \frac{|z|^{1-\lambda}}{2\Gamma(2-\lambda)}(2-|z|)\leqq |D_z^{\lambda}f(z)|\leqq\frac{|z|^{1-\lambda}}{2\Gamma(2-\lambda)}(2+|z|)$$

for $0<\lambda<1$ *and* $z\in\mathcal{U}$.

Proof. Since $f(z)\in\mathcal{P}$, (3.1) implies that $F(z)\in\mathcal{P}$. Therefore, by using Lemma 3, we get

$$(3.6)\qquad 2\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}|a_n| \leqq \sum_{n=2}^{\infty}\frac{n\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}|a_n|\leqq 1.$$

This shows that

$$(3.7)\qquad |F(z)|=|\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}f(z)| \leqq |z|+|z|^2\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}|a_n| \leqq |z|+\frac{1}{2}|z|^2$$

which implies the right hand side of (3.4).

Similarly we have the left hand side of (3.4).

Finally the second half of theorem follows from $G(z)\in\mathcal{P}$.

COROLLARY 2. *Under the hypotheses of* Theorem 3, $D_z^{-\lambda}f(z)$ *is included in a disk with its center at the origin and radius* $3/2\Gamma(2+\lambda)$, *and* $D_z^{\lambda}f(z)$ *is included in a disk with its center at the origin and radius* $3/2\Gamma\ (2-\lambda)$.

THEOREM 4. *If the function $f(z)$ defined by* (1.1) *in the class* $\mathcal{P}^*(\alpha)$, *then*

$$(3.8)\qquad \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left(1-\frac{1-\alpha}{2-\alpha}|z|\right)\leqq|D_z^{-\lambda}f(z)|$$
$$\leqq\frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left(1-\frac{1-\alpha}{2-\alpha}|z|\right)$$

for $\lambda>0$ *and* $z\in\mathcal{U}$, *and*

$$(3.9)\qquad \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1-\frac{2(1-\alpha)}{2-\alpha}|z|\right\}\leqq|D_z^{\lambda}f(z)|$$
$$\leqq\frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1+\frac{2(1-\alpha)}{2-\alpha}|z|\right\}$$

for $0<\lambda<1$ *and* $z\in\mathcal{U}$.

Proof. Since $f(z)\in\mathcal{T}^*(\alpha)$, (2.1) gives that

$$(3.10)\qquad \sum_{n=2}^{\infty}|a_n|\leqq\frac{1-\alpha}{2-\alpha},$$

Note that

$$(3.11)\qquad 0<\frac{\Gamma(n+1)\Gamma(2+\lambda)}{\Gamma(n+1+\lambda)}<1$$

for $n\geqq2$ and $\lambda>0$. Hence, with the aid of (3.10), we obtain

$$(3.12)\qquad |F(z)|=|\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}f(z)|$$
$$\leqq|z|+|z|^2\sum_{n=2}^{\infty}|a_n|$$
$$\leqq|z|+\frac{1-\alpha}{2-\alpha}|z|^2$$

and

$$(3.13)\qquad |F(z)|=|\Gamma(2+\lambda)z^{-\lambda}D_z^{-\lambda}f(z)|$$
$$\geqq|z|-|z|^2\sum_{n=2}^{\infty}|a_n|$$
$$\geqq|z|-\frac{1-\alpha}{2-\alpha}|z|^2$$

for $\lambda>0$ and $z\in\mathcal{U}$. Thus we can see that (3.4) follows from (3.12) and (3.13).

Next, it is easy that

$$(3.14)\qquad \frac{2-\alpha}{2}\sum_{n=2}^{\infty}n|a_n|\leqq\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leqq1-\alpha$$

and

$$1 < \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} < n \tag{3. 15}$$

for $n \geqq 2$ and $0 < \lambda < 1$. Consequently, by using (3. 14) and (3. 15), we get

$$\begin{aligned} |G(z)| &= |\Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z)| \\ &\leqq |z| + |z|^2 \sum_{n=2}^{\infty} n |a_n| \\ &\leqq |z| + \frac{2(1-\alpha)}{2-\alpha} |z|^2 \end{aligned} \tag{3. 16}$$

and

$$\begin{aligned} |G(z)| &= |\Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z)| \\ &\geqq |z| - |z|^2 \sum_{n=2}^{\infty} n |a_n| \\ &\geqq |z| - \frac{2(1-\alpha)}{2-\alpha} |z|^2 \end{aligned} \tag{3. 17}$$

which shows (3. 5). Thus we have the theorem.

COROLLARY 3. *Under the hypotheses of* Theorem 4, $D_z^{-\lambda} f(z)$ *is included in a disk with its center at the origin and radius* $(3-2\alpha)/(2-\alpha)\Gamma(2+\lambda)$, *and* $D_z^{\lambda} f(z)$ *is included in a disk with its center at the origin and radius* $(4-3\alpha)/(2-\alpha)\Gamma(2-\lambda)$.

THEOREM 5. *If the function* $f(z)$ *defined by* (1. 1) *is in the class* $\mathcal{Q}(\alpha)$, *then*

$$\begin{aligned} \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 - \frac{1-\alpha}{2(2-\alpha)} |z|\right\} &\leqq |D_z^{-\lambda} f(z)| \\ &\leqq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{1 + \frac{1-\alpha}{2(2-\alpha)} |z|\right\} \end{aligned} \tag{3. 18}$$

for $\lambda > 0$ *and* $z \in \mathcal{U}$, *and*

$$\begin{aligned} \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 - \frac{1-\alpha}{2-\alpha} |z|\right) &\leqq |D_z^{\lambda} f(z)| \\ &\leqq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left(1 + \frac{1-\alpha}{2-\alpha} |z|\right) \end{aligned} \tag{3. 19}$$

for $0 < \lambda < 1$ *and* $z \in \mathcal{U}$.

The proof of Theorem 5 is obtained by using the same technique as in the proof of Theorem 4 with the aid of (2. 5).

COROLLARY 4. *Under the hypotheses of* Theorem 5, $D_z^{-\lambda} f(z)$ *is included in a disk with its center at the origin and radius* $(5-3\alpha)/2(2-\alpha)\Gamma(2+\lambda)$, *and* $D_z^{\lambda} f(z)$ *is included in a disk with its center at the origin and radius*

$(3-2\alpha)/(2-\alpha)\Gamma(2-\lambda)$.

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Department of Mathematics
Kinki University
Osaka, Japan