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STRICTLY CYCLIC BANACH REPRESENTATIONS NAIMARK-RELATED TO *-REPRESENTATIONS AND THEY ARE *-REPRESENTATIONS

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1. Introduction.

In The first part of this note auther studied strictly cyclic Banach representations of Banach *-algebra (or C^*-algebra) Naimark-related to *-representations which is the weaker than the similarity problem, while in the second part the auther considered when strictly cyclic Banach representations of a C^*-algebra equal to *-representations, this is the stronger property than the similarity. Throughout this note X will denotes a complex Banach space, $B(X)$ the algebra of all bounded linear operators on X. A rpresentation π is always continuous and not *-preserving. A representation $\pi : A \to B(X)$ is said to be essential if for any $\xi \in X$ the condition $\pi(x)\xi=0$ for all $x \in A$ implies $\xi=0$. It is called strictly cyclic (resp. topologically cyclic) if there is a vector ξ_0 in X such that $\pi(A)\xi_0=X$ (resp. $\overline{\pi(A)\xi_0}=X$). A representation is called topologically irreducible provided that the only closed π-invariant subspaces of X are $\{0\}$ and X, and it is algebraically irreducible if the only π-invariant subspaces of X are $\{0\}$ and X.

2. Naimark-related problem for representations of a Banach *-algebra.

DEFINITION 2.1. Let $\pi : A \to B(X)$, $\psi : A \to B(X)$ be Banach representations of a Banach *-algebra A. π and ψ are said to be Naimark-related if there exists a closed densely defined one-to-one linear operator U defined in X with dense range in Y such that

(i) domain of U is π-invariant
(ii) $\psi(x) U\xi = U\pi(x)\xi$ for any $\xi \in$ domain of U and any $x \in A$.

THEOREM 2.2. *Let A be a Banach *-algebra, let $\pi : A \to B(X)$ be a strictly cyclic Banach representation with a strictly cyclic vector ξ_0 in X. If there*

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exists a state f on A with the following conditions:

(i) $f(x^*x)^{1/2} \leq K\|\pi(x)\xi_0\|$ $(K>0)$ *for all* $x\in A$

and (ii) $\{x\in A : \pi(x)\xi_0=0\} = \{x\in A : f(x^*x)=0\}$.

Then π *is Naimark–related to a *–representation.*

Proof. We define an inner product $<,>$ in $X=\pi(A)\xi_0$ by

$$f(y^*x)=<\pi(x)\xi_0,\ \pi(y)\xi_0> \quad (x, y\in A).$$

According to a property of a state $f(y^*x)\leq f(y^*y)f(x^*x)$ and the condition (ii), it is easily shown that the inner product is well defined with the property that

(1) $$<\pi(z)\pi(x)\xi_0,\ \pi(y)\xi_0>=<\pi(x)\xi_0,\ \pi(z^*)\pi(y)\xi_0>$$

for all x, y and z in A.

We denote the inner product space $(\pi(A)\xi_0, < >)$ by H_0. It is known that every *–representation of a U^*–algebra A on a pre–Hilbert space is continuous and each $\pi(z)$ $(z\in A)$ is also continuous on the pre–Hilbert space. Furthermore, every complex Banach *–algebra is a U^*–algebra ([8], proposition 5, Theorem 7). Thus $\pi(z)\in B(H_0)$ $(z\in A)$ and $\pi : A\rightarrow B(H_0)$ is continuous. Let H be the completion of H_0 and we define an operator extension of $\pi(z)$ $(z\in A)$ by

(2) $$\overline{\pi(z)\xi}=\lim_{\xi_n\rightarrow\xi} \pi(z)\xi_n, \quad \{\xi_n\}\subset H_0 \text{ and } \xi\in H.$$

Then this extension is unique with the properties that

(3) $$\|\overline{\pi(z)}\|=\|\pi(z)\|, \quad \overline{\pi(z)}\,|\,H_0=\pi(z) \quad (z\in A).$$

And also $\overline{\pi(z)}$ is a closed operator, whence $(\overline{\pi(z)})^*=\pi(z)^*\in B(H)$.

Since $\pi(z)^*$ is a closed operator whether $\pi(z)$ is closed or not, thus $\overline{\pi(z)}^* =\pi(z)^*$. It follows that

(4) $$\overline{\pi(z_1z_2)}\xi=\overline{\pi(z_1)\pi(z_2)}\xi,$$

$$(\overline{\pi(z)})^*\xi=\pi(z)^*\xi=\overline{\pi(z)^*\xi}=\overline{\pi(z^*)}\xi \quad (\xi\in H).$$

Now, we define $\bar{\pi} : A\rightarrow B(H)$ by

(5) $$\bar{\pi}(z)=\overline{\pi(z)} \quad (z\in A).$$

Then $\bar{\pi}$ is the continuous *–representation of A on the Hilbert space H.

It remains to prove that π and $\bar{\pi}$ are Naimark–related. We define a linear operator $U : X\rightarrow H_0$ by $U\xi=\xi$ $(\xi\in X)$, then the domain U is X, range of

U is dense in H and U is one-to-one. For each $\xi\in X$, there is a sequence $\{\xi_n\}$ in X such that $U\xi_n=\xi_n\to\xi$ for X-norm. But by (i)

$$<\pi(x_n)\xi_0-\xi,\ \pi(x_n)\xi_0-\xi>=f((x_n-x)^*(x_n-x))\leq K^2\|\pi(x_n)\xi_0-\xi\|^2.$$

This shows that X-norm limit can be con sidered as an H-norm limit. Thus if $\xi_n\to\xi$ and $U\xi_n\to\xi$ for the X-norm, then $\xi\in X$(=domain of U) and $U\xi=\xi\in H_0\subset H$. Therefore U is a closed operator.

Furthermore, since

$$\bar{\pi}(x)U\xi=\pi(x)\xi=U\pi(x)\xi \quad \text{for any} \quad \xi\in X \ (x\in A),$$

π and $\bar{\pi}$ are Naimark-related. We have proved the Theorem.

In special case, if the equality $f(x^*x)^{1/2}=\|\pi(x)\xi_0\|$ $(x\in A)$ holds in condition (i) of the Theorem 2.2, we have a following stronger result.

PROPOSITION 2.3. *Let A be a Banach *-algebra, let π be a Banach representation of A on a Banach space X with a striclty cyclic vector ξ_0 in X. If there exist a state f on A such that*

(i′) $$f(x^*x)^{1/2}=\|\pi(x)\xi_0\| \ (x\in A),$$

*then X will be a Hilbert space in the same norm as the initial one, and π is a *-representation of A on H.*

REMARK. (a) Let A be a Banach *-algebra. A positive linear form f on A is said to be a state if 1 is the minimum value of m such that

$$|f(x)|^2\leq mf(x^*x) \ \text{for all} \ x\in A.$$

(b) If A is a C^*-algebra with unit e ($\|e\|=1$), then a positive linear form f on A is a state if and only if $\|f\|=1$.

For the proof, let $f: A\to\mathbf{C}$ be a state, then

$$|f(x)|^2\leq mf(x^*x)\leq m\|f\|\|x^*x\|=m\|f\|\|x\|^2,$$

then

$$|f(\frac{x}{\|x\|})|^2\leq m\|f\| \quad (x\neq 0), \ \text{i.e.} \ \|f\|\leq m.$$

Hence, if $\min\{m: |f(x)|^2\leq mf(x^*x), x\in A\}=1$ then $\|f\|=1$.

Conversely, suppose that $\|f\|=1$. Since

$$|f(x)|^2\leq f(e)f(x^*x)\leq\|f\|\|e\|f(x^*x)=f(x^*x) \quad (x\in A)$$

we have $\min\{m: |f(x)|^2\leq mf(x^*x), \ x\in A\}=1$.

A subset $\{x\in A: f(x^*x)=0\}$ of A is said to be the left kernel of the state f.

THEOREM 2.4. *Let A be a C^*-algebra with unit, let $\pi : A \to B(X)$ be a strictly cyclic Banach representation with a strictly cyclic vector ξ_0 in X. If a subset $\{x \in A : \pi(x)\xi_0 = 0\}$ is a left kernel of a state f of A, then π is Naimark-related to a *-representation $\bar{\pi}$. ($\bar{\pi}$ is the *-representation defined in 2.2).*

Proof. We put $J = \{x \in A : \pi(x)\xi_0 = 0\}$. J is a closed left ideal of A. Define a map $\phi : A/J \to \pi(A)\xi_0 = X$ by $\phi(x+J) = \pi(x)\xi_0$, $x \in A$. Then ϕ is a bijection and continuous; the continuity from the quotient space A/J follows from the fact that

$$\|\phi(x+J)\| = \|\pi(x+z)\xi_0\|, \quad z \in J. \leq \|\pi\|\|x+z\|\|\xi_0\|, \quad z \in J.$$

Hence $\|\phi(x+J)\| \leq \|\pi\|\|\xi_0\|\|x+J\|_Q$ $(x \in A)$, where $\|\cdot\|_Q$ is the quotient norm on A/J. Therefore ϕ is a homeomorphism by the open mapping theorem, whence there exist $m > 0$ such that

$$\|\phi(x+J)\| = \|\pi(x)\xi_0\| \geq m\|x+J\|_Q \quad (x \in A).$$

Since A is a C^*-algebra, $\|f\| = 1$ so we have following inequality

$$f(x^*x) = \inf_{z \in J} f((x+z)^*(x+z)) \leq \inf_{z \in J}\|x+z\|^2 = \|x+J\|^2{}_Q.$$

It follows that

$$f(x^*x) \leq \|x+J\|^2{}_Q \leq \frac{1}{m^2}\|\pi(x)\xi_0\|^2$$

$$\text{i.e. } f(x^*x)^{1/2} \leq K\|\pi(x)\xi_0\|, \quad \left(k = \frac{1}{m}\right).$$

Therefore, by Theorem 2.2, the conclusion follows.

LEMMA 2.5. *Let A be a C^*-algebra with unit, J a closed left ideal of A such that $A/J = \{x+J : x \in A\}$ is separable for the quotient norm. Then there exists a state f on A such that*

$$J = \{x \in A : f(x^*x) = 0\}.$$

Proof. Let $A^+ = \{x^*x : x \in A\}$, then the set $A^+ \cap J$ is a face in the positive cone A^+. For any $x \in A/J$, $x^*x \geq 0$ and $x^*x \notin J$. Applying the geometric form of Hahn-Banach theorem to the fact $A^+ \cap J$, we obtain a state f on A such that

$$f = 0 \text{ on } J, \quad f(x^*x) > 0 \quad (x \in A) \text{ and } f(e) = 1.$$

Since A/J is separable, there exists a sequence $\{x_n\}$ in A such that $f_n|J = 0$, $f_n(x_n{}^*x_n) > 0$ and $f_n(e) = 1$ $(n = 1, 2, \ldots)$.

We put $f=\sum_{n=1}^{\infty}\frac{1}{2^n}f_n$, then obviously f is a normalized state on A with $f|J=0$. Moreover since J is a left ideal and $|f(y)|^2\leq f(y^*y)f(e)$ holds for any state f on A,

$$f(y^*y)=0 \Longleftrightarrow y^*y\in J \Longleftrightarrow y\in J.$$

$$\text{i. e. } J=\{x\in A : f(x^*x)=0\}.$$

From the Theorem 2.4 and the Lemma 2.5, we have a following

COROLLARY 2.6. *Let A be a C^*-algebra with unit, let $\pi : A\to B(X)$ be a strictly cyclic Banach representation. If A/J is separable for the quotient norm, then π is Naimark-related to a *-representation, where $J=\{x\in A : \pi(x)\xi_0=0\}$ and ξ_0 is a strictly cyclic vector of π.*

Also we have the following result:

COROLLARY 2.7. *Let A be a separable C^*-algebra with unit, let $\pi : A\to B(X)$ a strictly cyclic Banach representation. Then π is Naimark-related to a *-representation of A on some Hilbert space.*

2.8. EXAMPLE. We illustrate that the conditions (i) , (ii) of the Theorem 2.2 are resonable.

Let Δ be the set of all complex homomorphism of a commutative Banach algebra A with identity. The formula $\hat{x}(h)=h(x)$ $(h\in\Delta,\ x\in A)$ assigns to each $x\in A$ a function $\hat{x} : \Delta\to\mathbf{C}$. We call $\hat{x}$ the Gelfand transformation of x. Let $\hat{A}=\{\hat{x} : x\in A\}$. The Gelfand topology of Δ is the weak topology induced by $\hat{A}$. Obviously $\hat{A}\subset C(\Delta)$, $C(\Delta)$ is the algebra of complex continuous functions on Δ. The set Δ equipped with the Gelfand topology is called the maximal ideal space of A; this is a compact Hausdorff space.

The Gelfand-Naimark theorem says that if A is commutative C^*-algebra with the maximal ideal space Δ, then the Gelfand transform $\Lambda : A\to\hat{A}\subset C(\Delta)$ is an isometric isomorphism of A onto $C(\Delta)$ which has the property that $(x^*)^\wedge=\overline{\hat{x}}$.

We define a representation $\pi : A\to B(A)$ of a commutative C^*-algebra with unit and with the maximal ideal space Δ on A by $\pi(x)y=xy$. Then π is a continuous Banach representation with a strictly cyclic vector e (unit of A).

Also we define a linear functional $f : A\to\mathbf{C}$ by $f(x)=\int_\Delta\hat{x}d\mu$, where μ is a positive regular Borel measure on Δ such that $\mu(\Delta)\leq 1$, supp $\mu=\Delta$.

Then $f(y^*x)=\int_\Delta\hat{x}\hat{y}^*d\mu=\int_\Delta\hat{x}\overline{\hat{y}}d\mu$

and $f(x^*x)=\int_\Delta|x|^2d\mu=0 \Longleftrightarrow \hat{x}=0 \Longleftrightarrow x=0$

hold by the Gelfand–Naimark theorem. Therefore

$$\{x\in A : \pi(x)e=0\} = \{x\in A : f(x^*x)=0\} = \{0\},$$

and we can define an inner product $\langle , \rangle$ in A by

$$\langle x, y\rangle = f(y^*x) \quad (x, y\in A).$$

Furthermore,

$$f(x^*x)^{1/2} = (\int_\Delta |\hat{x}(h)|^2 d(h))^{1/2} \leq \max_{h\in\Delta} |\hat{x}(h)| = \|\hat{x}\|_\infty$$

$$= \|x\| = \|\pi(x)e\| \quad (x\in A).$$

i. e. $f(x^*x)^{1/2} \leq \|\pi(x)e\|$ for all $x\in A$.

In this case the extension $\bar{\pi}$ of π is defined by $\overline{\pi(x)y} = \lim_{y_n\to y} \pi(x)y_n = xy$ $(y\in H)$.

If $\pi : A\to B(X)$ is an essential algebraically irreducible Banach representation of a Banach *-algebra A on X, then every nonzero vector ξ in X is strictly cyclic and the set $\{x\in A : \pi(x)\xi=0\}$ is a maximal modular left ideal.

Thus if there is a state f on A such that $\{x\in A : \pi(x)\xi=0\}$ is the left kernel of the state f, then π is Naimark-related to a *-representation. The converse of this statement is true ([4], p. 9, Theorem 1).

3. Conditions for a strictly cyclic Banach representation to be a *-representation.

Let A be a Banach *-algebra and f a state on A. f is said to be pure if every state on A dominated by f is of the form λf $(0\leq\lambda\leq 1)$.

3. 1. THEOREM. *Let A be a C^*-algebra, $\pi : A\to B(X)$ a strictly cyclic Banach representation with a strictly cyclic vector ξ_0 in X. If there exists a pure state f on A such that $\{x\in A : \pi(x)\xi_0=0\}$ is the left kernel of f, then X is a Hilbert space in an equivalent norm and π is a *-representation of A on this Hilbert space.*

Proof. $M=\{x\in A : f(x^*x)=0\}$ is a closed left ideal of A, we define an inner product on $A/M=\{x+M : x\in A\}$ by

$$(x+M, \ y+M) = f(y^*x) \quad (x, y\in A).$$

Then $|x+M| = f(x^*x)^{1/2}$ is a norm of $x+M\in A/M$. This norm coincides with the quotient norm $\|x+M\|_Q = \inf\{\|x+z\| : z\in M\}$ on A/M ([16], Theorem 1). Thus A/M is a Hilbert space for the norm $\|\cdot\|_Q$. Define a map $\phi : A/M$

$\to\pi(A)\xi_0=X$ by $\phi(x+M)=\pi(x)\xi_0$ $(x\in A)$, then ϕ is bijection and continuous. Therefore ϕ is a homeomorphism by the open mapping theorem, whence two norms $\|\cdot\|_Q$ and the original norm of X are equivalent. Now, we define a new norm $\|\cdot\|_2$ of X by $\|\pi(x)\xi_0\|_2=\|x+M\|_Q$ $(=f(x^*x)^{1/2})$. Then this is the induced norm by an inner product on X, that is, $<\pi(x)\xi_0,\ \pi(y)\xi_0>=f(y^*x)$. And, by definition, the norm $\|\pi(x)\xi_0\|_2=f(x^*x)^{1/2}$ on $\pi(A)\xi_0=X$ is a Hilbert space norm. Obviously two norms $\|\cdot\|_2$ and the original norm $\|\cdot\|$ on X are equivalent. We put $(X,\|\cdot\|_2)=H$, then by proof of the Theorem 2.2, $\bar{\pi}=\pi: A\to B(H)$ is a *-representation.

3.2. LEMMA. *Let A be a Banach algebra with unit e, $\|e\|=1$. Let K be a proper left ideal of A. Then there exists a maximal left ideal M containing K.*

Method of the proof is used a standard tool, we omit the proof. We have also a following Lemma.

3.3. LEMMA. *Let A be a Banach algebra. A maximal left ideal M of A is closed.*

3.4. THEOREM. *Let A be a C^*-algebra with unit e, let $\pi: A\to B(X)$ be a topologically irreducible Banach representation. If there is a strictly cyclic vector ξ_0 of π in X, then X is a Hilberts pace H in an equivalent norm and π is a *-representation of A on H.*

Proof. We put $K=\{x\in A:\pi(x)\xi_0=0\}$, then K is a closed left ideal of A. We show that K is maximal. By Lemma 3.2 and 3.3, there exists a msximal left ideal M of A such that $K\subseteq M=\overline{M}$. Let A/K be a quotient space equipped with the quotient norm. A map $\phi: A/K\to\pi(A)\xi_0=X$, defined by $\phi(x+K)=\pi(x)\xi_0$ $(x\in A)$, is a homeomorphism.

Since M is closed in A, M/K is closed in A/K. Thus $\phi(M/K)=\pi(M)\xi_0$ is closed in X since ϕ is a homeomorphism.

And since

$$\pi(x)\phi(M/K)=\pi(x)\pi(M)\xi_0\subseteq\pi(M)\xi_0=\phi(M/K)\quad(x\in A),$$

$\phi(M/K)$ is a closed π-invariant subspace of X. Hence we have

$$\phi(M/K)=X=\pi(A)\xi_0=\phi(A/K).$$

i.e. $M/K=A/K$ or $M=A$, whence $M=K$.

Therefore, there exists a state f on A such that

$$K=\{x\in A: f(x^*x)=0\},\ \|f\|=f(e)=1\quad([13],\ \text{p. }259,\ \text{p. }266).$$

This state f is also a pure state ([1], p. 463).

It follows that there is a pure state f on A such that

$$\{x \in A : \pi(x)\xi_0 = 0\} = \{x \in A : f(x^*x) = 0\}.$$

Thus, by Theorem 3.1, the conclusion follows.

The condition on existence of strictly cyclic vector ξ_0 in Theorem 3.4 may be replaced by another condition. In general we have a following fact: Let X be a locally convex topological vector space over $\mathbf{K}$($\mathbf{K}=C$ or $\mathbf{R}$) for a topology τ, let S be a convex subset of X, then S is τ-closed if and only if it is weakly closed ([14], p. 158).

3.5. COROLLARY. *Let A be a C^*-algebra with unit, let $\pi : A \to B(X)$ be topologically irreducible Banach representation. If there is a vector ξ_0 in X and there exists a left ideal J of A such that $0 \neq \pi(J)\xi_0$ is weakly closed in X, then the same conclusion follows as the* Theroem 3.4.

For a proof, since $\pi(J)\xi_0$ is a linear subspace of X, it is convex. By the above statement $\pi(J)\xi_0$ is weakly closed if and only if it is norm closed in X. Thus $\pi(A)\xi_0 = X$. Thus the result follows.

Now we are going to proove the following.

3.6. THEOREM. *Let A be a C^*-algebra with unit, $\pi : A \to B(X)$ a Banach representation with a strictly cyclic vector ξ_0 of π.*

If there exists a irreducible state f on A such that $\{x \in A : \pi(x)\xi_0 = 0\}$ is the left kernel of this state f on A, then X is a Hilbert space in an equivalent norm, f is a strictly pure state on A and π is an algebraically irreducible $*$-representation of A on this Hilbert space.

In order to prove this theorem we have to add some definitions and preliminary subjects.

A positive sesquilinear form (PSE) ϕ on a $*$-algebra A said to be adjunctive if $\phi(xy, z) = \phi(y, x^*z)$ $(x, y, z \in A)$.

If f is a state on a $*$-algebra A, then the formula

$$\phi_f(x, y) = f(y^*x) \quad (x, y \in A)$$

defined an adjunctive PSF ϕ_f.

Let ϕ be an adjunctive PSF on a $*$-algebra A and if

$$N_\phi = \{x \in A : \phi(x, x) = 0\}$$

then N_ϕ is a left ideal of A. The quotient vector space $A_\phi = A/N_\phi$ is an inner product space with inner product defined by

$$(x_\phi | y_\phi) = \phi(x, y), \text{ where } x_\phi = x + N_\phi \ (x \in A).$$

For each $a\in A$, $x_\psi\to(ax)_\psi$ is a well defined linear map on the quotient vector space A_ψ; We denote it by the formula $T_a x_\psi=(ax)_\psi$. Then this satisfies $(T_a x_\psi | y_\psi)=(x_\psi | T_{a*} y_\psi)$.

A positive sesquilinear form ψ on a *-algebra A is said to be admissible if (i) ψ is adjunctive and (ii) for each $a\in A$ there exists a constant $K_a\geq 0$ such that $\psi(ax, ax)\leq K_a\psi(x, x)$ for all $x\in A$.

If ψ is admissible *PSF* on a *-algebra A then $T_a: A_\psi\to A_\psi$ is continuous linear map on the inner product space A_ψ. The unique continuous extension of T_a to the completion H_ψ of A_ψ will denote also by T_a, thus $T_a\in B(H_\psi)$.

Let A be a *-algebra. A representation $a\to T_a\in B(H_\psi)$ is said to be *-representation determined by a admissible *PSF* ψ on A.

An admissible *PSF* ψ on a *-algebra A is said to be strictly irreducible if $H_\psi\neq 0$ and if the *-subset $\{T_a : a\in A\}$ of $B(H_\psi)$ is striclty irreducible.

A state f on a *-algebra A is admissible if the adjunctive *PSF* ψ_f determined by $\psi_f(x, y)=f(y^*x)$ is admissible, and a state f is strictly irreducible if the *-subset $\{T_a\in B(H_{\psi_f}) | a\in A\}$ is strictly irreducible.

We denote $H_f=H_{\psi_f}$, thus $N_f=\{x\in A : f(x^*x)=0\}$ and $A_f=A/N_f$.

Let A be a Banach *-agebra. A positive form f on A is called strictly pure if f is pure and the *-representation of A determined by f (or ψ_f) is strictly irreducible.

Proof of the Theorem 3.6. Since on a C^*-algebra with unit every state f is continuous and f is admissible ([14], p. 293). Therefore f is strictly irreducible if and only if f is a pure state on A ([14], Theorem 67.22), in this case f is also a strictly pure state ([1], p. 460, Theorem 2.1).

Hence, by assumptions and Theorem 3.1, X is a Hilbert space H in an equivalent norm and π is a *-representation of A on H.

We put $N_f=\{x\in A : f(x^*x)=0\}$, $A_f=A/N_f$ and $|x_f|=|x+N_f|=f(x^*x)^{1/2}$ $(x\in A)$. This norm $|\cdot|$ on A_f is a complete norm and the map $\phi : A_f\to\pi(A)\xi_0=X$ defined by $\phi(x_f)=\pi(x)\xi_0(x_f\in A_f)$ is a homeomorphism. And the *-subset $\{Ta : a\in A\}\subset B(A_f)(=B(H_f))$ is strictly irreducible by the hypothesis on f, where T_a is defined by $T_a x_f=(ax)_f$, $x_f\in A_f$.

Moreover since $\phi(T_a x_f)=\phi(ax+N_f)=\pi(a)\pi(x)\xi_0$, we have

$$\phi(T_a x_f)=\pi(a)\phi(x_f) \quad (a\in A,\ x_f\in A_f).$$

It follows that the *-subset $\{T_a : a\in A\}$ of $B(H_f)$ is strictly irreducible if and only if π is algebriaically irreducible; for, if $\pi(a)S\subset S$ for all $a\in A$ then there exists a unique subset M of A_f such that $\phi(M)=S$. Thus

$$\pi(a)\phi(M)\subset\phi(M)\Longleftrightarrow\phi(T_a M)\subset\phi(M)\Longleftrightarrow T_a M\subset M$$

for all $a \in A$, and obviously

$$M = N_f \text{ or } A_f \text{ if and only if } S = \{0\} \text{ or } X.$$

Therefore $\pi : A \rightarrow B(H_f)$ is an algebraically irreducible *-representation of A on H_f, we put $H_f = H$, this completes the proof.

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