J. Korean Math. Soc.
Vol. 20, No. 2, 1983

A SPECIAL SUBCLASS OF CLOSE TO CONVEX FUNCTIONS

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1. Introduction.

Let S denote the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in the unit disk $E=\{z:|z|<1\}$. A function $\phi\in S$ is said to be a convex function, denoted $\phi\in K$, if

$$\mathrm{Re}\left\{1+z\frac{\phi''(z)}{\phi'(z)}\right\}>0,\quad (z\in E).$$

The subclass C of S which consists of close-to-convex functions f is characterized by

$$(1.1)\qquad \mathrm{Re}\left\{\frac{f'(z)}{\phi'(z)}\right\}>0,\quad z\in E$$

for some $\phi\in K$.

Let V be the family of functions $f\in S$ satisfying the condition

$$(1.2)\qquad \mathrm{Re}\{(1-z^2)f'(z)\}>0$$

for all $z\in E$. It is known [2] that $f\in V$ is univalent and $f(E)$ is a domain convex in the direction of imaginary axis, that is, the intersection of $f(E)$ with each vertical line is connected, or empty. The class V has been, further, studied in [3] and [4].

Let $V(\alpha,\beta)$ be the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, analytic in E, and satisfying in E the condition

$$(1.3)\qquad \left|\frac{(1-z^2)f'(z)-1}{2\beta((1-z^2)f'(z)-\alpha)-((1-z^2)f'(z)-1)}\right|<1$$

for some $\alpha,\beta\,(0\leq\alpha<1,\ 0<\beta\leq1)$. Clearly, $V(0,1)\equiv V$. Further, by fixing $\alpha=0$, $\beta=1$ in (1.3) and on choosing $\phi(z)=\frac{1}{2}\log\{(1+z)/(1-z)\}$ in (1.1), we observe that (1.1) coincides with (1.3). In fact, functions satisfy-

Received January 31, 1983
This research was supported by a University of Khartoum research grant, and was completed while visitting at I. I. T., Kanpur, India.

ing (1. 3), we may call, special close-to-convex functions of order α and type β associated with $\phi(z)=\frac{1}{2}\log\ \{(1+z)/(1-z)\}$.

We observe that by special choices of α, β our class, $V(\alpha, \beta)$, gives rise to various new subclasses of S; for instance,

$$V^*\equiv V(0, \tfrac{1}{2})=\{f\in S : |(1-z^2)f'(z)-1|<1, z\in E\},$$

$$V_\delta^*\equiv V(0, (2\delta-1)/(2\delta))=\{f\in S : |(1-z^2)f'(z)-\delta|<\delta, \delta>\tfrac{1}{2}, z\in E\},$$

$$V(\rho)\equiv V((1-\rho)/(1+\rho), (1+\rho)/2)$$
$$=\{f\in S : |((1-z^2)f'(z)-1)/((1-z^2)f'(z)+1)|<\rho, 0<\rho\leq 1, \ z\in E\},$$

$$V^*(\alpha)\equiv V(0, 1-\alpha)=\{f\in S : |(1-z^2)f'(z)-1/(2\alpha)|<1/(2\alpha),$$
$$0\leq\alpha<1, z\in E\}.$$

Since the class $V(\alpha, \beta)$ includes various subclasses of S, a study of its convexity properties will lead to a unified study of these classes. In this paper, we determine the radii of convexity for the class $V(\alpha, \beta)$. In particular, we deduce the corresponding results for the classes V^*, V_δ^*, $V(\rho)$ and $V^*(\alpha)$.

2. The Radii of Convexity for $V(\alpha, \beta)$.

THEOREM. *Let $f\in V(\alpha, \beta)$. Let r_0 be the smallest positive root in $(0, 1]$ of the equation*

$$(2.1)\quad (2\beta-1)(2\alpha\beta-1)(r-2)r^3-2(\beta+\alpha\beta+2\alpha\beta^2-1)r^2-2r+1=0.$$

Then

(i) for $0\leq r<r_0$, f is convex in $|z|<r_1$, where r_1 is the smallest positive root in $(0, 1)$ of the equation

$$(2.2)\quad (2\beta-1)(1-2\alpha\beta)r^4+2(1-2\beta)r^3+2\beta(2\alpha\beta-\alpha-1)r^2$$
$$+2(2\alpha\beta-1)r+1=0;$$

(ii) and for $r_0\leq r<1$, f is convex in $|z|<r_2$ where r_2 is the smallest positive root in $(0, 1)$ of the equation

$$(2.3)\quad \alpha(1-\alpha)(2\beta-1)r^8-2\alpha\beta(1-\alpha)r^6+(2\alpha^2\beta-2\alpha\beta+4+2\alpha^2-6\alpha)r^4$$
$$+2\alpha\beta(1-\alpha)r^2-\alpha(1-\alpha)=0.$$

The above bounds are sharp.

Proof. Let $f \in V(\alpha, \beta)$; then the condition (1. 3) coupled with an application of Schwarz's lemma gives

$$(2.4) \qquad (1-z^2)f'(z) = \frac{1+(2\alpha\beta-1)w(z)}{1+(2\beta-2)w(z)},$$

where w is analytic function in E and satisfies the condition $w(0)=0$ and $|w(z)|<1$ for $z \in E$. Taking logarithmic derivative of (2. 4), we find that

$$(2.5) \qquad 1+z\frac{f''(z)}{f'(z)} = \frac{1+z^2}{1-z^2} - \frac{2\beta(1-\alpha)zw'(z)}{(1+(2\beta-1)w(z))(1+(2\alpha\beta-1)w(z))}.$$

Taking the real part of (2. 5) of both sides,

$$(2.6) \qquad \mathrm{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} = Re\left\{\frac{1+z^2}{1-z^2}\right\} - 2\beta(1-\alpha)$$
$$Re\left\{\frac{zw'(z)}{(1+(2\beta-1)w(z))(1+(2\alpha\beta-1)w(z))}\right\}.$$

Since $\phi(z)=w(z)/z$ is also a bounded analytic function in E, we have by [5, p. 168]

$$\left|\frac{zw'(z)-w(z)}{z^2}\right| = |\phi'(z)| \le \frac{r^2-|w(z)|^2}{r^2(1-r^2)}, \quad (|z|=r)$$

or

$$(2.7) \qquad |zw'(z)-w(z)| \le \frac{r^2-|w(z)|^2}{1-r^2}.$$

Equation (2. 6) yields in conjunction with (2. 7),

$$(2.8) \qquad \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \ge \frac{1-r^2}{1+r^2} + \frac{1}{2\beta(1-\alpha)}\left\{\mathrm{Re}\left((2\beta-1)p(z)+\frac{(2\alpha\beta-1)}{p(z)}\right)\right.$$
$$\left. - \frac{r^2|(2\beta-1)p(z)-(2\alpha\beta-1)|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\right\} - \frac{\beta+\alpha\beta-1}{\beta(1-\alpha)},$$

where

$$(2.9) \qquad p(z) = \frac{1+(2\alpha\beta-1)w(z)}{1+(2\beta-1)w(z)}.$$

We observe that the transformation (2. 9) maps the disk $|w(z)| \le r$ onto the disk $|w(z)-a|<d$, where

$$a = \frac{1-(2\beta-1)(2\alpha\beta-1)r^2}{1-(2\beta-1)^2r^2}, \quad d = \frac{2\beta(1-\alpha)r}{1-(2\beta-1)^2r^2}.$$

Setting $p(z)=a+u+iv$, $R=|p(z)|$ and taking $M(u, v)$ as the expression on the right hand side of (2. 8), we get

$$(2.10)\quad M(u,v)=\frac{1-r^2}{1+r^2}+\frac{1}{2\beta(1-\alpha)}\{(2\beta-1)(a+u)-2(\beta+\alpha\beta-1)$$
$$-\frac{1-(2\beta-1)^2r^2}{1-r^2}\ \frac{d^2-u^2-v^2}{R}+\frac{(2\alpha\beta-1)(a+u)}{R^2}\Big\}.$$

On differentiating (2. 10) partially with respect to v,

$$\frac{\partial M(u,v)}{\partial v}=\frac{vR^4N(u,v)}{2\beta(1-\alpha)}$$

where

$$N(u,v)=2(1-2\alpha\beta)(a+u)+\frac{(1-(2\beta-1)^2r^2)(d^2-u^2-v^2)R}{1-r^2}$$
$$+\frac{2(1-(2\beta-1)^2r^2)R^3}{1-r^2}.$$

It is easily seen that $N(u,v)>0$ and so the minimum of $M(u,v)$ on every chord u=constant is attained when $v=0$. Thus the minimum of $M(u,v)$ inside the disk $|p(z)-a|\le d$ is attained on the diameter $v=0$. Setting $v=0$ in (2. 10), we obtain

$$L(R)\equiv M(u,0)=\frac{1-r^2}{1+r^2}+\frac{1}{\beta(1-\alpha)(1-r^2)}\{\beta(1-(2\beta-1)r^2)R$$
$$+\alpha\beta(1-(2\alpha\beta-1)r^2)R^{-1}-a(1-2\beta-1)^2r^2\}-$$
$$\frac{\beta+\alpha\beta-1}{\beta(1-\alpha)}$$

for $a-d\le R\le a+d$. Thus it follows that absolute minimum of $L(R)$ in $(0,\infty)$ is attained at

$$(2.11)\qquad R_0=\left\{\frac{\alpha(1-(2\alpha\beta-1)r^2}{1-(2\beta-1)r^2}\right\}^{\frac{1}{2}},$$

and equals

$$(2.12)\quad L(R_0)=\frac{1-r^2}{1+r^2}+\frac{1}{(1-\alpha)(1-r^2)}$$
$$\{\sqrt{4\alpha(1-(2\beta-1)r^2L(1-(2\alpha\beta-1)r^2)}$$
$$-(1+\alpha)+(4\alpha\beta-\alpha-1)r^2\}.$$

We note that $R_0<a+d$. However, R_0 may not always be greater than $a-d$. Hence the minimum of $L(R)$ is attained at

$$(2.13)\qquad R_1=a-d=\frac{1+(2\alpha\beta-1)r}{1+(2\beta-1)r},$$

and is equal to

$$(2.14)\qquad L(R_1)=\frac{1-r^2}{1+r^2}-\frac{2\beta(1-\alpha)r}{(1+(2\beta-1)r)(1+(2\alpha\beta-1)r)}.$$

Therefore, from (2.12) and (2.14) we get

$$(2.15)\qquad \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}$$

$$\geq\begin{cases}\dfrac{1+2(2\alpha\beta-1)r+2\beta(2\alpha\beta-\alpha-1)r^2-2(2\beta-1)r^3-(2\beta-1)(2\alpha\beta-1)r^4}{(1+r^2)(1+(2\beta-1)r)(1+(2\alpha\beta-1)r)}; \\ \text{for } R_0\leq R_1, \\ \dfrac{1-r^2}{1+r^2}+\dfrac{\mu(r,\alpha,\beta)}{(1-\alpha)(1-r^2)}, \quad \text{for } R_0\geq R_1\end{cases}$$

where

$$(2.16)\qquad \mu(r,\alpha,\beta)=\sqrt{4\alpha(1-(2\beta-1)r^2)(1-(2\alpha\beta-1)r^2)} \\ -(1+\alpha)+(4\alpha\beta-\alpha-1)r^2.$$

Hence the radii of convexity of $f(z)$ are determined by the following equations:

$$(2.17)\qquad 1+2(2\alpha\beta-1)r+2\beta(2\alpha\beta-\alpha-1)r^2 \\ -2(2\beta-1)r^3-(2\beta-1)(2\alpha\beta-1)r^4=0 \ \text{for } R_0\leq R_1,$$

$$(2.18)\qquad \frac{1-r^2}{1+r^2}+\frac{\mu(r,\alpha,\beta)}{(1-\alpha)(1-r^2)}=0, \ \text{or } R_0\geq R_1,$$

where $\mu(r,\alpha,\beta)$ is given in (2.16). After some computations, the expressions (2.17) and (2.18), reduce, respectively, to the equations (2.2) and (2.3). Also the two minima given by (2.15) become equal to each other for such values of α,β for which $R_0=R_1$ or (2.1).

The extremal functions for two inequalities in Theorem are attained, respectively, for the functions given by

$$(2.19)\qquad f_1(z)=\frac{1+(2\beta-1)(2\alpha\beta-1)}{2(1+(2\beta-1)^2)}\cdot\log\frac{1+z}{1-z}+\frac{i\beta(1-\alpha)}{1+(2\beta-1)^2} \\ \log\frac{(1-(2\beta-1)iz)^2}{1-z^2}$$

at $z=ir$, and

$$f_2'(z)=\frac{1+2i\alpha\beta bz-(2\alpha\beta-1)z^2}{(1+2i\beta bz-(2\beta-1)z^2)(1-z^2)} \text{ at } z=ir, \tag{2.20}$$

where b is determined by the equation

$$\frac{1-2\alpha\beta br+(2\alpha\beta-1)r^2}{1-2\beta br+(2\beta-1)r^2}=R_0=\left\{\frac{\alpha(1-(2\alpha\beta-1)r^2}{1-(2\beta-1)r^2}\right\}^{\frac{1}{2}}$$

3. Applications

By fixing $\alpha=0$, $\beta=1$ in Theorem we get the following result, which was obtained earlier in [3].

COROLLARY 1. *Let $f\in V$. Then $f(z)$ maps the disk $|z|<R$ onto a convex domain where R is the smallest positive root of the equation*

$$r^4-r^3-2r^2-2r+1=0.$$

We note that $R=\frac{1}{2}(1+\sqrt{5})-\left\{\frac{1}{2}(1+\sqrt{5})\right\}^{\frac{1}{2}}=0.346\cdots$ *The bound is sharp.*

COROLLARY 2. *Let $f\in V(\alpha, 1)$. Let α_0 denote the smallest positive root of the equation*

$$80\alpha^5-244\alpha^4+266\alpha^3-62\alpha^2-17\alpha+14=0. \tag{3.1}$$

Then

(i) *for $0\leq\alpha\leq\alpha_0$, f is convex in $|z|<r_1$, where r_1 is the smallest positive root of the equation*

$$(1-2\alpha)r^4-2r^3-2(1-\alpha)r^2+2(2\alpha-1)r+1=0; \tag{3.2}$$

(ii) *for $\alpha_0\leq\alpha<1$, f is convex in $|z|<r_2^2$, where r_2 is the smallest positive root of the equation*

$$\alpha x^4-2\alpha x^3+4(1-\alpha)x^2+2\alpha x-\alpha=0. \tag{3.3}$$

All these bounds are sharp.

Proof. Taking $\beta=1$ in (2.1) we get

$$(2\alpha-1)r^3-3(2\alpha-1)r^2-3r+1=0. \tag{3.4}$$

Putting $\beta=1$ in (2.2) and (2.3) we obtain, respectively (3.2) and (3.3). By eliminating r between (3.2) and (3.4) we get (3.1).

We observe that (3.2) coincides with a result in [1, Theorem 2.4(i)] but (3.3) does not coincide with Theorem 2.4 (ii) in [1]; perhaps because of computational errors in [1].

Taking $\alpha=0$ and $\beta=\frac{1}{2}$ in above theorem we obtain

COROLLARY 3. *Let* $f\in V^*$; *then* $f(z)$ *maps the disk* $|z|<\sqrt{2}-1$ *onto convex domain. The bound is sharp.*

By replacing (α,β), *respectively, by* $(0,1-\alpha)$, $(0,(2\delta-1)/2\delta)$, *and* $((1-\rho)/(1+\rho),(1+\rho)/2)$ *in Theorem, we get the following results.*

COROLLARY 4. *Let* $f\in V^*(\alpha)$, $(0\leq\alpha<1)$. *Then* $f(z)$ *maps the disk* $|z|<R$ *onto a convex domain, where* R *is the smallest positive root in* $[0,1]$ *of the equation*

$$(1-2\alpha)(r-2)r^3-2(1-\alpha)r^2-2r+1=0.$$

The bound is sharp.

COROLLARY 5. *Let* $f\in V_\delta^*$, $(\delta>\frac{1}{2})$. *Then* $f(z)$ *maps the disk* $|z|<R$ *onto a convex domain, where* R *is the smallest positive root in* $[0,1]$ *of the equation*

$$D(r-2)r^3-(D+1)r^2-2r+1=0,$$

where $D=1-1/\delta$. *The bound is sharp.*

COROLLARY 6. *Let* $f\in V(\rho)$, $(0<\rho\leq1)$. *Let* r_0 *be the smallest positive root in* $[0,1]$ *of the equation*

$$\rho^2(2-r)r^3-(1-\rho^2)r^2-2r+1=0.$$

Then

(i) *for* $0\leq r<r_0$, $f(z)$ *is convex in* $|z|<r_1$, *where* r_1 *is the smallest positive root of the equation*

$$\rho^2r^4-2\rho r^3-(1+\rho^2)r^2-2\rho r+1=0;$$

(ii) *for* $r_0\leq r<1$, $f(z)$ *is convex in* $|z|<r_2^2$, *where* r_2 *is the smallest positive root of the equation*

$$2\rho^2(1-\rho)x^4-2\rho(1-\rho^2)x^3+2\rho(1+6\rho+\rho^2)x^2+2\rho(1-\rho^2)x-2\rho(1-\rho)=0.$$

The above bounds are sharp.

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