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INFINITESIMAL VARIATIONS OF SUBMANIFOLDS OF $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$

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0. Introduction

Recently the present authors [4] studied infinitesimal variations of invariant submanifolds with normal (f, g, u, v, λ)-structure. Yano and Kon [3] studied infinitesimal variations of an even dimensional sphere.

The purpose of the present paper is to study infinitesimal variations of submanifolds of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

In §1, we state some of known results on structures [1] which $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ admits.

In §2, we investigate infinitesimal variations of various kinds of submanifolds.

In §3, we study infinitesimal variations of f-invariant submanifold and k-invariant one.

§4 is devoted to the study of isometric variation and f-preserving variation of a f-invariant and k-invariant submanifold.

And in §5, on a compact f-invariant and k-invariant submanifold with induced (f, g, u, v, λ)-structure, we investigate some variation-preserving relations.

The last §6 is devoted to the study of infinitesimal variations of f-antiinvariant submanifold and k-antiinvariant one.

1. Preliminaries

Let $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ be a submanifold of codimension 2 of $(2n+2)$ dimensional Euclidean space E^{2n+2} and be covered by a system of coordinate neighbourhoods $\{U, x^h\}$, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, 3, \cdots, 2n\}$. ($S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is the direct product differentiable manifold of two spheres $S^n(1/\sqrt{2})$ with radius $1/\sqrt{2}$ and with its center at the origin in E^{n+1}.)

Let denote by Z, C and D the position vector of a point of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$, the first unit normal vector in the direction opposite to that of

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the radius vector of $S^{2n+1}(1)$ and the second unit normal vector in the direction normal to $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ and tangent to $S^{2n+1}(1)$ respectively.

In E^{2n+2}, there exists a natural Käehlerian structure

$$F=\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}, \tag{1.1}$$

E being the unit square matrix of order $n+1$. Of course, F satisfies

$$F^2=-I,\quad FU\cdot FV=U\cdot V \tag{1.2}$$

for arbitrary vectors U and V in E^{2n+2}, I and $\cdot$ denoting the identity transformation in E^{2n+2} and the inner product of two vectors in Euclidean space respectively.

Now we put

$$Z_i=\partial Z/\partial x^i,\quad g_{ji}=Z_j\cdot Z_i, \tag{1.3}$$

where Z_i are $2n$ linearly independent vectors of E^{2n+2} tangent to $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

Applying F to Z_i, C and D gives

$$FZ_i=f_i{}^hZ_h+u_iC+v_iD, \tag{1.4}$$

$$FC=-u^iZ_i+\lambda D, \tag{1.5}$$

$$FD=-v^iZ_i-\lambda C, \tag{1.6}$$

where $f_i{}^h$ are the components of a tensor field of type $(1,1)$, u_i and v_i are the components of 1-forms, λ is a function on $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$, u_i and v_i are respectively given by $u^i=u_jg^{ji}$ and $v^i=v_jg^{ji}$, g^{ih} being contravariant components of the metric tensor g_{ji}.

From (1.2), (1.4), (1.5) and (1.6), we find

$$\begin{cases} f_j{}^if_i{}^h=-\delta_j^h+u_ju^h+v_jv^h, \\ u_if_j{}^i=\lambda v_j,\quad f_i{}^hu^i=-\lambda v^h, \\ v_if_j{}^i=-\lambda u_j,\quad f_i{}^hv^i=\lambda u^h, \\ u_iu^i=v_iv^i=1-\lambda^2,\quad u_iv^i=0, \\ f_j{}^mf_i{}^lg_{ml}=g_{ji}-u_ju_i-v_jv_i. \end{cases} \tag{1.7}$$

A set of f, g, u, v and λ satisfying these equations is called an (f, g, u, v, λ)-structure. It is verified that $f_{ji}=f_j{}^lg_{li}$ is skew-symmetric in j and i.

Now applying the operator ∇_j of covariant differentiation with respect to the Riemannian connection to (1.4), (1.5) and (1.6), and taking account of $\nabla_j F=0$, we find

$$(1.8)\qquad \begin{cases} \nabla_j f_i^{\ h}=-g_{ji}u^h+\delta_j^h u_i-k_{ji}v^h+k_j^{\ h}v_i,\\ \nabla_j u_i=f_{ji}-\lambda k_{ji},\quad \nabla_j v_i=-k_{jl}f_i^{\ l}+\lambda g_{ji},\\ \nabla_j\lambda=-2v_j, \end{cases}$$

where k_{ji} are the components of the second fundamental tensors with respect to the second unit normal D. From (1.7) and the last equation of (1.8), λ does not vanish almost everywhere in $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$. Moreover $k_j^{\ h}$ are given by the following form [1]:

$$(1.9)\qquad (k_j^{\ h})=\begin{pmatrix} E & 0\\ 0 & -E\end{pmatrix},$$

where E being the unit square matrix of order n and $k_j^{\ h}=k_{ji}g^{ih}$. From (1.4) and (1.9), we find [1]

$$(1.10)\qquad k_j^{\ i}u^j=-v^i,\quad k_j^{\ i}v^j=-u^i.$$

We have from the last two equations of (1.8)

$$(1.11)\qquad k_m^{\ h}f_i^{\ m}+f_m^{\ h}k_i^{\ m}=0,$$

that is, $k_m^{\ h}$ and $f_i^{\ m}$ are anticommute with each other.

Let M^m be an m-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{V, y^a\}$ and isometrically immersed in $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ by the immersion $i: M^m\to S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$, where here and in the sequel the indices $a, b, c\cdots$ run over the range $\{1, 2, 3, \cdots, m\}$. We identify $i(M^m)$ with M^m and represent the local expression of the immersion i by $x^h=x^h(y^a)$. If we put $B_b^{\ h}=\partial_b x^h$, $(\partial_b=\partial/\partial y^b)$, then $B_b^{\ h}$ are m linearly independent vectors of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ tangent to M^m. Denoting by g_{cb} the Riemannian metric tensor of M^m, we have $g_{cb}=g_{ji}B_c^{\ j}B_b^{\ i}$ since the immersion is isometric. We denote by $C_y^{\ h}$ $2n-m$ mutually orthogonal unit normals to M^m, where here and in the sequel the indices x, y, z run over the range $\{m+1, m+2, m+3, \cdots, 2n\}$.

The transforms $f_i^{\ h}B_b^{\ i}$ and $k_i^{\ h}B_b^{\ i}$ of $B_b^{\ i}$ by $f_i^{\ h}$ and $k_i^{\ h}$ are written in the form respectively

$$(1.12)\qquad f_i^{\ h}B_b^{\ i}=f_b^{\ a}B_a^{\ h}-f_b^{\ x}C_x^{\ h},$$

$$(1.13)\qquad k_i^{\ h}B_b^{\ i}=k_b^{\ a}B_a^{\ h}+k_b^{\ x}C_x^{\ h}.$$

and the transforms $f_i^{\ h}C_y^{\ i}$ and $k_i^{\ h}C_y^{\ i}$ of $C_y^{\ i}$ by $f_i^{\ h}$ and $k_i^{\ h}$ in the form

$$(1.14)\qquad f_i^{\ h}C_y^{\ i}=f_y^{\ a}B_a^{\ h}+f_y^{\ x}C_x^{\ h},$$

(1.15) $$k_i^h C_y^i = k_y^a B_a^h + k_y^x C_x^h.$$

From (1.12)～(1.15), we have

(1.16) $$\begin{cases} f_{ba} = -f_{ab}, \ f_{bx} = f_{xb}, \ f_{xy} = -f_{yx}, \\ k_{ba} = k_{ab}, \ k_{ax} = k_{xa}, \ k_{xy} = k_{yx}, \end{cases}$$

where $f_{by} = f_b^z g_{zy}$, $f_{yb} = f_y^c g_{cb}$, $k_{by} = k_b^z g_{zy}$ and $k_{yb} = k_y^c g_{cb}$, g_{zy} being the metric tensor of the normal bundle of M^m. From (1.11)～(1.15), we have

(1.17) $$f_b^a k_a^c + k_b^a f_a^c = f_b^y k_y^c - k_b^y f_y^c,$$

(1.18) $$f_b^a k_a^x - f_b^y k_y^x = k_b^a f_a^x - k_b^y f_y^x,$$

(1.19) $$f_x^a k_a^y + f_x^z k_z^y = k_x^a f_a^y - k_x^z f_z^y.$$

We put

(1.20) $$u^h = B_a^h u^a + C_x^h u^x, \ v^h = B_a^h v^a + C_x^h v^x.$$

We get from (1.10), (1.13), (1.15) and (1.20)

(1.21) $$\begin{cases} u^a = -v^b k_b^a - v^x k_x^a, \ v^a = -u^b k_b^a - u^x k_x^a, \\ u^x = -v^a k_a^x - v^y k_y^x, \ v^x = -u^a k_a^x - u^y k_y^x. \end{cases}$$

When $f_i^h B_b^i$ and $k_i^h B_b^i$ are always tangent to M^m respectively, that is, when $f_b^x = 0$ and $k_b^x = 0$, M^m is said to be *f-invariant* and *k-invariant* respectively. In order for M^m to be f-invariant and k-invariant respectively, it is necessary and sufficient that

(1.22) $$\begin{cases} f_{bx} = f_b^y g_{yx} = -f_{ji} B_b^j C_x^i = 0, \\ k_{bx} = k_b^y g_{yx} = k_{ji} B_b^j C_x^i = 0 \end{cases}$$

respectively.

When $f_i^h B_b^i$ and $k_i^h B_b^i$ are always normal to M^m respectively, that is, $f_b^a = 0$ and $k_b^a = 0$, M^m is said to be *f-antiinvariant* and *k-antiinvariant* respectively. In order for M^m to be f-antiinvariant and k-antiinvariant respectively, it is necessary and sufficient that

(1.23) $$\begin{cases} f_{ba} = f_b^c g_{ca} = f_{ji} B_b^j B_a^i = 0, \\ k_{ba} = k_b^c g_{ca} = k_{ji} B_b^j B_a^i = 0 \end{cases}$$

respectively.

Equations of Gauss and those of Weingarten for M^m are respectively written as

(1.24) $$\nabla_c B_b^h = h_{cb}^x C_x^h$$

and

(1.25) $$\nabla_c C_y{}^h=-h_c{}^a{}_y B_a{}^h,$$

∇_c denoting the van der Waerden-Bortollotti covariant differentiation along M^m, where $h_{cb}{}^x$ are the second fundamental tensors of M^m with respect to $C_x{}^h$, $h_c{}^a{}_y=h_{cby}g^{ba}$ and $h_{cby}=h_{cb}{}^z g_{zy}$.

2. Infinitesimal variations of submanifolds of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$

We now consider an infinitesimal variation

(2.1) $$\bar{x}^h=x^h+\xi^h(y)\varepsilon$$

of a submanifold M^m of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$, where ξ^h is a vector field of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ defined along M^m and ε is an infinitesimal. We then have $\bar{B}_b{}^h=B_b{}^h+\partial_b\xi^h\varepsilon$, where $\bar{B}_b{}^h=\partial_b\bar{x}^h$ are m linearly independent vectors tangent to the varied submanifold at varied point $(\bar{x}^h)$. We displace $\bar{B}_b{}^h$ back parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h) and obtain $\tilde{B}_b{}^h=B_b{}^h+(\nabla_b\xi^h)\varepsilon$, neglecting terms of order higher than one with respect to ε. In the sequel, we neglect always terms of order higher than one with respect to ε. Thus putting $\delta B_b{}^h=\tilde{B}_b{}^h-B_b{}^h$, we obtain

(2.2) $$\delta B_b{}^h=(\nabla_b\xi^h)\varepsilon.$$

On the other hand, putting

(2.3) $$\xi^h=\xi^a B_a{}^h+\xi^x C_x{}^h,$$

we have

(2.4) $$\nabla_b\xi^h=(\nabla_b\xi^a-h_b{}^a{}_x\xi^x)B_a{}^h+(\nabla_b\xi^x+h_{ba}{}^x\xi^a)C_x{}^h.$$

When the tangent space at a point (x^h) of the submanifold and that at the corresponding $(\bar{x}^h)$ of the varied submanifold are parallel, the variation is said to be *parallel*.

From (2.2) and (2.4) we see that in order for an infinitesimal variation to be parallel, it is necessary and sufficient that

(2.5) $$\nabla_b\xi^x+h_{ba}{}^x\xi^a=0.$$

We denote by $\bar{C}_y{}^h$ $2n-m$ mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}_y{}^h$ vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $\bar{C}_y{}^h$ from the varied point $(\bar{x}^h)$ back to the original point (x^h). Putting $\delta C_y{}^h=\tilde{C}_y{}^h-C_y{}^h$, we find [2]

(2.6) $$\bar{C}_y{}^h=C_y{}^h-\Gamma^h{}_{ji}\xi^j C_y{}^i\varepsilon+\delta C_y{}^h,$$

where Γ_{ji}^h are Christoffel symbols formed with g_{ji}. Assuming that $\delta C_y{}^h$ are infinitesimals of order one with respect to ε and putting

$$\delta C_y{}^h = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon, \tag{2.7}$$

we have

$$\eta_y{}^a = -(\nabla^a \xi_y + h_b{}^a{}_y \xi^b), \tag{2.8}$$

where $\xi_y = \xi^x g_{yx}$. The $\eta_y{}^x$ appearing in (2.7) is a tensor field of the normal bundle of M^m satisfying $\eta_{yx} + \eta_{xy} = 0$, η_{yx} being defined by $\eta_{yx} = \eta_y{}^z g_{zx}$.

(1) *Infinitesimal variations of submanifolds of* $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ *tangent to* u^h.

Suppose that a submanifold M^m of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ is tangent to u^h. Then we have equation of the form $B_a{}^h u^a = u^h$, from which, differentiating covariantly along M^m and using (1.8), (1.12) and (1.13), we find

$$h_{cb}{}^x u^b C_x{}^h + B_a{}^h (\nabla_c u^a) = (f_c{}^a - \lambda k_c{}^a) B_a{}^h - (f_c{}^x + \lambda k_c{}^x) C_x{}^h,$$

and consequently

$$\nabla_c u^a = f_c{}^a - \lambda k_c{}^a, \quad h_{cby} u^b = -f_{cy} - \lambda k_{cy}. \tag{2.9}$$

In order that the varied submanifold is also tangent to $u^h(\bar{x})$, it is necessary and sufficient that we have equation of the form $\bar{B}_a{}^h \bar{u}^a = u^h(\bar{x})$, from which, putting $\bar{u}^a = u^a + \delta u^a$,

$$(B_b{}^h + \partial_b \xi^h \varepsilon)(u^b + \delta u^b) = u^h + \xi^j \partial_j u^h \varepsilon.$$

Thus using (1.8), we find

$$B_b{}^h \delta u^b + [(\nabla_b \xi^h) u^b - f_i{}^h \xi^i + \lambda k_j{}^h \xi^j]\varepsilon = 0. \tag{2.10}$$

On the other hand, using (1.12)$\sim$(1.15) and (2.3), we find

$$\begin{cases} f_i{}^h \xi^i = (f_b{}^a \xi^b + f_y{}^a \xi^y) B_a{}^h + (-f_b{}^x \xi^b + f_y{}^x \xi^y) C_x{}^h, \\ k_i{}^h \xi^i = (\xi^e k_e{}^a + \xi^x k_x{}^a) B_a{}^h + (\xi^e k_e{}^x + \xi^y k_y{}^x) C_x{}^h. \end{cases} \tag{2.11}$$

Thus, substituting (2.4) and (2.11) into (2.10), we find

$$\begin{aligned} B_a{}^h \delta u^a &+ [(\nabla_b \xi^a - h_b{}^a{}_x \xi^x) u^b - (f_b{}^a \xi^b + f_y{}^a \xi^y) \\ &+ \lambda(\xi^b k_b{}^a + \xi^x k_x{}^a)] B_a{}^h \varepsilon \\ &+ [(\nabla_b \xi^x + h_{be}{}^x \xi^e) u^b + (f_b{}^x \xi^b - f_y{}^x \xi^y) \\ &+ \lambda(\xi^a k_a{}^x + \xi^y k_y{}^x)] C_x{}^h \varepsilon = 0, \end{aligned}$$

or, using (2.9),

$$B_a{}^h\delta u^a+(u^b\nabla_b\xi^a-\xi^b\nabla_b u^a+2\lambda\xi^x k_x{}^a)B_a{}^h\varepsilon$$
$$+(u^b\nabla_b\xi^x-f_y{}^x\xi^y+\lambda\xi^y k_y{}^x)C_x{}^h\varepsilon=0,$$

from which

$$\delta u^a=(\mathcal{L}u^a-2\lambda\xi^x k_x{}^a)\varepsilon, \tag{2.12}$$

$\mathcal{L}$ denoting the Lie derivative with respect to ξ^a and

$$u^b\nabla_b\xi^x-f_y{}^x\xi^y+\lambda\xi^y k_y{}^x=0. \tag{2.13}$$

Thus we have

PROPOSITION 2. 1. *In order for an infinitesimal variation* (2. 1) *to carry a submanifold M^m of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ tangent to u^h into a submanifold $\bar{M}^m$ also tangent to $u^h(\bar{x})$, it is necessary and sufficient that* (2. 13) *holds, the variation of u^a being given by* (2. 12).

(2) *Infinitesimal variations of a submanifolds of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ normal to u^h.*

Suppose that a submanifold M^m of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is normal to u^h. Then we have $B_b{}^i u_i=0$, from which, differentiating covariantly along M^m and using (1. 8), (1. 12), (1. 13) and (1. 20), we find

$$h_{cb}{}^x u_x+B_b{}^i(f_c{}^a B_{ai}-f_c{}^x C_{xi})-\lambda k_{cb}=0,$$

where $B_{ai}=B_a{}^j g_{ji}$ *and* $C_{xi}=C_x{}^j g_{ji}$ and consequently

$$h_{cb}{}^x u_x+f_{cb}-\lambda k_{cb}=0.$$

Thus, $h_{cb}{}^x u_x$ and λk_{cb} being symmetric and f_{cb} being skew symmetric in c and b, we have

$$h_{cb}{}^x u_x=\lambda k_{cb},\quad f_{cb}=0. \tag{2.14}$$

Thus we have

PROPOSITION 2. 2. *If a submanifold M^m of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is normal to u^h, then M^m is f-antiinvariant.*

Now in order that the varied submanifold is also normal to $u^h(\bar{x})$, it is necessary and sufficient that we have $\bar{B}_b{}^i u_i(\bar{x})=0$, from which

$$[B_b{}^i+(\partial_b\xi^i)\varepsilon]\ [u_i+\xi^j\partial_j u_i\varepsilon]=0.$$

Thus, using (1. 8), we find

$$(\nabla_b\xi^i)u_i+f_{ji}\xi^j B_b{}^i-\lambda k_{ji}B_b{}^j\xi^i=0.$$

Substituting (2. 4) and (2. 11) into this equation, we have

$$(\nabla_b\xi^x+h_{ba}{}^x\xi^a)u_x+f_{cb}\xi^c+f_{yb}\xi^y-\lambda\xi^a k_{ab}-\lambda\xi^x k_{xb}=0,$$

from which, using (2.14),

$$(\nabla_b\xi^x)u_x+f_{yb}\xi^y-\lambda\xi^x k_{xb}-\lambda\xi^x k_{xb}=0. \tag{2.15}$$

Thus we have

PROPOSITION 2.3. *In order for an infinitesimal variation* (2.1) *to carry a submanifold* M^m *of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *normal to* u^h *into a submanifold normal to* $u^h(\bar{x})$, *it is necessary and sufficient that* (2.15) *holds.*

(3) *Infinitesimal variatons of submanifolds of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *tangent to* v^h.

Suppose that a submanifold M^m of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is tangent to v^h. Then we have equation of the form $B_a{}^h v^a=v^h$, from which, differentiating covariantly along M^m and using (1.8) and (1.12)∼(1.14), we have

$$\begin{cases}\nabla_c v^a=k_c{}^e f_e{}^a+k_c{}^x f_x{}^a+\lambda\delta_c^a,\\ h_{ca}{}^x v^a=-k_c{}^a f_a{}^x+k_c{}^y f_y{}^x.\end{cases} \tag{2.16}$$

Now, in order that the varied submanifold is also tangent to $v^h(\bar{x})$, it is necessary and sufficient that we have equation of the form $\bar{B}_a{}^h\bar{v}^a=v^h(\bar{x})$, from which, putting $\bar{v}^a=v^a+\delta v^a$,

$$(B_b{}^h+\partial_b\xi^h\varepsilon)\ \ (v^b+\delta v^b)=v^h+\xi^j(\partial_j v^h)\varepsilon.$$

Thus, using (1.8), we have

$$B_a{}^h\delta v^a+[(\nabla_b\xi^h)v^b-\lambda\xi^h+\xi^j k_{ji}f^{hi}]\varepsilon=0.$$

Substituting (2.3), (2.4), (1.12), (1.14) and (2.11) into this equation, we find

$$\begin{aligned}B_a{}^h\delta v^a&+[(\nabla_b\xi^a-h_b{}^a{}_x\xi^x)v^b-\lambda\xi^a-(\xi^c k_c{}^b+\xi^x k_x{}^b)f_b{}^a\\ &-(\xi^c k_c{}^x+\xi^y k_y{}^x)f_x{}^a]B_a{}^h\varepsilon\\ &+[(\nabla_b\xi^x+h_{ba}{}^x\xi^a)v^b-\lambda\xi^x+(\xi^c k_c{}^b+\xi^y k_y{}^b)f_b{}^x\\ &-(\xi^c k_c{}^y+\xi^z k_z{}^y)f_y{}^x]C_x{}^h\varepsilon=0,\end{aligned}$$

from which, using (2.16) and (1.18),

$$\delta v^a=[\pounds v^a+2(k_x{}^b f_b{}^a+k_x{}^y f_y{}^a)\xi^x]\varepsilon, \tag{2.17}$$

$\pounds$ denoting the Lie derivative with respect to ξ^a and

$$v^b\nabla_b\xi^x-\lambda\xi^x+(k_y{}^a f_a{}^x-k_y{}^z f_z{}^x)\xi^y=0. \tag{2.18}$$

Thus we have

PROPOSITION 2. 4. *In order for an infinitesimal variation* (2. 1) *to carry a submanifold* M^m *of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *tangent to* v^h *into a submanifold tangent to* $v^h(\bar{x})$, *it is necessary and sufficient that* (2. 18) *holds, the variation of* v^a *being given by* (2. 17).

(4) *Infinitesimal variations of submanifolds of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *normal to* v^h.

Suppose that a submanifold M^m of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is normal to v^h. Then we have equation of the form $B_b{}^iv_i=0$ or $B_b{}^i\nabla_i\lambda=0$, which shows that λ is constant along M^m. Differentiating $B_b{}^iv_i=0$ covariantly along M^m and using (1. 8), (1. 12) and (1. 13), we find

(2. 19) $$h_{cb}{}^xv_x+\lambda g_{cb}-f_b{}^ak_{ac}+f_b{}^xk_x{}^c=0.$$

Now, in order that the varied submanifold is also normal to $v^h(\bar{x})$, it is necessary and sufficient that we have $\bar{B}_b{}^iv_i(\bar{x})=0$, from which

$$(B_b{}^i+(\partial_b\xi^i)\varepsilon)\ \ (v_i+(\xi^j\partial_jv_i)\varepsilon)=0,$$

Thus using (1. 8), (1, 12), (2. 3) and (2. 11), we have

$$(\nabla_b\xi^i)v_i+f_{cb}(\xi^ak_a{}^c+\xi^xk_x{}^c)+f_{xb}(\xi^ak_a{}^x+\xi^yk_y{}^x)+\lambda\xi_b=0.$$

Therefore substituting (2. 4) into this equation, we have

$$(\nabla_b\xi^x+h_{be}{}^x\xi^e)v_x+f_{cb}(\xi^ak_a{}^c+\xi^xk_x{}^c)+f_{xb}(\xi^ak_a{}^x+\xi^yk_y{}^x)+\lambda\xi_b=0,$$

or, using (2. 19)

(2. 20) $$(\nabla_b\xi^x)v_x+f_{cb}\xi^xk_x{}^c+f_{xb}\xi^yk_y{}^x=0.$$

Thus we have

PROPOSITION 2. 5. *In order for an infinitesimal variation* (2. 1) *to carry a submanifold* M^m *of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *normal to* v^h *into a submanifold normal to* $v^h(\bar{x})$, *it is necessary and sufficient that* (2. 20) *holds.*

When $\xi^x=0$, that is, the variation vector ξ^h is tangent to the submanifold, we say that the variation is *tangential* and when the variation vector ξ^h is normal to the submanifold, that is, $\xi^a=0$, we say that the variation is *normal*.

3. Infinitesimal variations of f-invariant submanifold and k-invariant one of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

We assume that M^m is an f-invariant submanifold of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$. We then have

$$f_i^hB_b^i=f_b^aB_a^h,\quad f_i^hC_y^i=f_y^xC_x^h. \tag{3.1}$$

Differentiating the first equation of (3.1) covariantly along M^m and using (1.8), (1.13) and (3.1), we find

$$\begin{aligned}&(-g_{cb}u^a+u_b\delta_c^a-k_{cb}v^a+v_bk_c^a)B_a^h\\&\qquad+(-g_{cb}u^x-k_{cb}v^x+v_bk_c^x+h_{cb}^yf_y^x)C_x^h\\&=(\nabla_cf_b^a)B_a^h+f_b^ah_{ca}^xC_x^h,\end{aligned}$$

and consequently, comparing the tangential and normal parts,

$$\nabla_cf_b^a=-g_{cb}u^a+u_b\delta_c^a-k_{cb}v^a+v_bk_c^a \tag{3.2}$$

and

$$h_{cb}^yf_y^x-h_{ce}^xf_b^e=g_{cb}u^x+k_{cb}v^x-v_bk_c^x, \tag{3.3}$$

from which, taking the skew symmetric part,

$$h_{ce}^xf_b^e-h_{be}^xf_c^e=v_bk_c^x-v_ck_b^x. \tag{3.4}$$

Differentiating the second equation of (3.1) covariantly along M^m and using (1.8), (1.13), (1.25) and (3.1), we find

$$\begin{aligned}&(u_y\delta_c^a-k_{cy}v^a+k_c^av_y-h_c{}^b{}_yf_b^a)B_a^h+(v_yk_c^x-k_{cy}v^x)C_x^h\\&=(\nabla_cf_y^x)C_x^h-h_c{}^a{}_xf_y^xB_a^h,\end{aligned}$$

and consequently

$$h_c{}^e{}_yf_e^a-h_c{}^a{}_xf_y^x=\delta_c^au_y-k_{cy}v^a+k_c^av_y, \tag{3.5}$$

which is equivalent to (3.3) and

$$\nabla_cf_y^x=v_yk_c^x-k_{cy}v^x. \tag{3.6}$$

We now consider an infinitesimal variation (2.1) and assume that it carries the f-invariant submanifold $M^m$of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ into an f-invariant submanifold. Then we have

$$f_i^h(x+\xi\varepsilon)\bar{B}_b^i=(f_b^a+\delta f_b^a)\bar{B}_a^h,$$

that is,

$$(f_i^h+\xi^j\partial_jf_i^h\varepsilon)\ (B_b^i+\partial_b\xi^i\varepsilon)=(f_b^a+\delta f_b^a)\ (B_a^h+\partial_a\xi^h\varepsilon),$$

from which, using (1.8), we obtain

$$\begin{aligned}&[f_i^h+\xi^j(-\Gamma_{jt}^hf_i^t+\Gamma_{ji}^tf_t^h-g_{ji}u^h+\delta_j^hu_i\\&\qquad-k_{ji}v^h+k_j^hv_i)\varepsilon]\ (B_b^i+\partial_b\xi^i\varepsilon)\end{aligned}$$

$$=(f_b{}^a+\delta f_b{}^a)(B_a{}^h+\partial_a\xi^h\varepsilon),$$

that is,

$$(f_i{}^h\nabla_b\xi^i-f_b{}^a\nabla_a\xi^h-\xi_b u^h+u_b\xi^h-k_{ji}v^h\xi^jB_b{}^i+k_j{}^hv_b\xi^j)\varepsilon=(\delta f_b{}^a)B_a{}^h.$$

Thus substituting (2.4), (1.20) and (2.11) into this equation, we have

$$\begin{aligned}&[(\nabla_b\xi^e-h_b{}^e{}_x\xi^x)f_e{}^a-f_b{}^e(\nabla_e\xi^a-h_e{}^a{}_x\xi^x)-\xi_b u^a+u_b\xi^a\\&\qquad-(\xi^ek_{eb}+\xi^xk_{xb})v^a+v_b(\xi^ek_e{}^a+\xi^xk_x{}^a)]B_a{}^h\varepsilon\\&+[(\nabla_b\xi^y+h_{be}{}^y\xi^e)f_y{}^x-f_b{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)-\xi_b u^x+u_b\xi^x\\&\qquad-(\xi^ek_{eb}+\xi^yk_{yb})v^x+v_b(\xi^ek_e{}^x+\xi^yk_y{}^x)]C_x{}^h\varepsilon\\&=(\delta f_b{}^a)B_a{}^h,\ \text{from which}\end{aligned}$$

$$\begin{aligned}\delta f_b{}^a=&[(\nabla_b\xi^e-h_b{}^e{}_x\xi^x)f_e{}^a-f_b{}^e(\nabla_e\xi^a-h_e{}^a{}_x\xi^x)-\xi_b u^a+u_b\xi^a\\&-(\xi^ek_{eb}+\xi^xk_{xb})v^a+v_b(\xi^ek_e{}^a+\xi^xk_x{}^a)]\varepsilon,\end{aligned}$$

or using (3.2) and (3.4),

(3.7) $$\delta f_b{}^a=[(\pounds f_b{}^a)+2f_b{}^eh_e{}^a{}_x\xi^x]\varepsilon$$

and

$$\begin{aligned}&(\nabla_b\xi^y+h_{be}{}^y\xi^e)f_y{}^x-f_b{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)-\xi_b u^x+u_b\xi^x\\&\qquad-(\xi^ek_{eb}+\xi^yk_{yb})v^x+v_b(\xi^ek_e{}^x+\xi^yk_y{}^x)=0,\end{aligned}$$

or, using (3.3),

(3.8) $$(\nabla_b\xi^y)f_y{}^x-f_b{}^e(\nabla_e\xi^x)+u_b\xi^x-\xi^yk_{yb}v^x+v_b\xi^yk_y{}^x=0.$$

Thus we have

THEOREM 3.1. *In order for an infinitesimal variation* (2.1) *to carry an f-invariant submanifold* M^m *of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *into an f-invariant one, it is necessary and sufficient that* (3.8) *holds, the variation of* $f_b{}^a$ *being given by* (3.7).

An infinitesimal variation given by (2.1) is called an *f-invariance preserving variation* if it carries an f-invariant submanifold into an f-invariant subnanifild. If an f-invariance preserving variation preserves $f_b{}^a$ then we say that it is *f-preserving*.

From (1.7), (3.1) and (1.20), we find

(3.9) $$f_b{}^cf_c{}^a=-\delta_b^a+u_bu^a+v_bv^a,$$

(3.10) $$f_c{}^ef_b{}^dg_{ed}=g_{cb}-u_cu_b-v_cv_b,$$

(3.11) $$f_b{}^au^b=-\lambda v^a,\quad f_b{}^av^b=\lambda u^a,$$

(3.12) $$u_a u^a = 1-\lambda^2 - u_x u^x, \quad v_a v^a = 1-\lambda^2 - v_x v^x,$$

(3.13) $$u_a v^a = -u_x v^x,$$

(3.14) $$u_x u_b = -v_x v_b,$$

(3.15) $$f_x{}^y f_y{}^z = -\delta_x^z + u_x u^z + v_x v^z,$$

(3.16) $$u^x f_{xy} = -\lambda v_y, \quad v^x f_{xy} = \lambda u_y.$$

Equations (3.9)~(3.13) show that a necessary and sufficient condition $f_b{}^a$, g_{cb}, u_b, v_b and λ to define (f, g, u, v, λ)-structure is that

(3.17) $$u_x = 0, \quad v_x = 0,$$

that is, the vector u^h and v^h are always tangent to the submanifold M^m.

Transvecting (3.4) with $f_d{}^b$, we get

(3.18) $$f_d{}^b f_c{}^e h_{be}{}^x = -h_{dc}{}^x + (u_d u^e + v_d v^e) h_{ce}{}^x + \lambda u_d k_c{}^x + v_c f_d{}^b k_b{}^x.$$

Now we assume that M^m is k-invariant submanifold of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

We then have

(3.19) $$k_i{}^h B_b{}^i = k_b{}^a B_a{}^h, \quad k_i{}^h C_y{}^i = k_y{}^x C_x{}^h.$$

Differentiating the first equation of (3.19) covariantly along M^m and using (1.9) and (3.19), we find

(3.20) $$\nabla_c k_b{}^a = 0, \quad h_{cb}{}^y k_y{}^x = k_b{}^a h_{ca}{}^x.$$

Differentiating the second equation of (3.19) covariantly along M^m and using (1.9) and (3.19), we have

(3.21) $$\nabla_c k_y{}^x = 0, \quad k_a{}^b h_{cy}{}^a = k_y{}^x h_c{}^b{}_x,$$

the second equation is equivalent to the second equation of (3.20). From (1.21), we have

(3.22) $$k_b{}^a u^b = -v^a, \quad k_b{}^a v^b = -u^a, \quad k_b{}^a k_a{}^c = \delta_b^c.$$

We now consider an infinitesimal variation (2.1) and assume that it carries the k-invariant submanifold M^m of $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$ into a k-invariant submanifold. Then we have

$$k_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i = (k_b{}^a + \delta k_b{}^a)\bar{B}_a{}^h,$$

that is,

$$(k_i{}^h + \xi^j \partial_j k_i{}^h \varepsilon)(B_b{}^i + \partial_b \xi^i \varepsilon) = (k_b{}^a + \delta k_b{}^a)(B_a{}^h + \partial_a \xi^h \varepsilon),$$

from which, using (1.9), we obtain

$$[k_i^h+\xi^j(-\Gamma_{jt}^h k_i^t+\Gamma_{ji}^t k_t^h)\varepsilon]\ (B_b^i+\partial_b\xi^i\varepsilon)$$
$$=(k_b^a+\delta k_b^a)(B_a^h+\partial_a\xi^h\varepsilon),$$

that is,

$$[k_i^h(\nabla_b\xi^i)-k_b^a\nabla_a\xi^h]\varepsilon=(\delta k_b^a)B_a^h.$$

Thus substituting (2.4) and (2.11) into this equation, we have

$$[(\nabla_b\xi^c-h_b{}^c{}_x\xi^x)k_c^a B_a^h+(\nabla_b\xi^y+h_{bc}{}^y\xi^c)k_y^x C_x^h$$
$$-(\nabla_c\xi^a-h_c{}^a{}_x\xi^x)k_b^c B_a^h-(\nabla_c\xi^x+h_{ce}{}^x\xi^e)k_b^c C_x^h]\varepsilon$$
$$=(\delta k_b^a)B_a^h,$$

from which,

$$\delta k_b^a=[(\nabla_b\xi^c-h_b{}^c{}_x\xi^x)k_c^a-(\nabla_c\xi^a-h_c{}^a{}_x\xi^x)k_b^c]\varepsilon$$

or, using (3.21),

$$\delta k_b^a=[(\nabla_b\xi^c)k_c^a-k_b^c\nabla_c\xi^a]\varepsilon \tag{3.23}$$

and

$$(\nabla_b\xi^y+h_{bc}{}^y\xi^c)k_y^x-k_b^c(\nabla_c\xi^x+h_{ce}{}^x\xi^e)=0, \tag{3.24}$$

or, using the second equation of (3.20),

$$k_y^x\nabla_b\xi^y-k_b^c\nabla_c\xi^x=0. \tag{3.25}$$

An infinitesimal variation given by (2.1) is called a *k–invariance preserving variation* if it carries a k–invariant submanifold into a k–invariant submanifold. Thus we have

THEOREM 3.2. *In order for an infinitesimal variation to be k–invariance preserving, it is necessary and sufficient that the variation vector* ξ^h *satisfies* (3.25), *the variation of* k_b^a *being given by* (3.23).

From (3.24) and (2.5), we have

THEOREM 3.3. *If the variation* (2.1) *of k–invariant submanifold is parallel, then it is k–invariance preserving.*

4. Isometric variation and f–preserving variation of an f–invariant and k-invariant submanifold with induced (f, g, u, v, λ)–structure.

Applying the operator δ to $g_{cb}=g_{ji}B_c^jB_b^i$ and using (2.2), (2.4) and $\delta g_{ji}=0$, we find [2]

$$\delta g_{cb}=(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\varepsilon, \tag{4.1}$$

from which,

(4. 2) $$\delta g^{ba}=-(\nabla^b\xi^a+\nabla^a\xi^b-2h^{ba}{}_x\xi^x)\varepsilon.$$

A variation of a submanifold for which $\delta g_{cb}=0$ is said to be *isometric.* By a straightforward computation, we obtain

(4. 3) $$\delta\Gamma_{cb}{}^a=[(\nabla_c\nabla_b\xi^h+K_{kji}{}^h\xi^k B_c{}^j B_b{}^i)B^a{}_h+h_{cb}{}^x(\nabla^a\xi_x+h_d{}^a{}_x\xi^d)]\varepsilon,$$

from which, using equations of Gauss and Codazzi of the submanifold M^m, we have

(4. 4) $$\delta\Gamma_{cb}{}^a=(\nabla_c\nabla_b\xi^a+K_{dcb}{}^a\xi^d)\varepsilon$$
$$-[\nabla_c(h_{bex}\xi^x)+\nabla_b(h_{cex}\xi^x)-\nabla_e(h_{cbx}\xi^x)]g^{ea}\varepsilon.$$

A variation of submanifold for which $\delta\Gamma_{cb}{}^a=0$ is said to be *affine.*

Suppose that the submanifold is f-invariant and k-invariant, and has induced (f, g, u, v, λ)-structure. Then, from (3. 18), (2. 9) and (2. 16), we have

(4. 5) $$f_d{}^b f_c{}^e h_{be}{}^x=-h_{dc}{}^x,\quad u^b h_{cbx}=v^b h_{cbx}=0.$$

If the variation of the submanifold is normal, we have from (3. 7) and (4. 1)

(4. 6) $$\delta f_b{}^a=(-f_e{}^a h_b{}^e{}_x\xi^x+f_b{}^e h_e{}^a{}_x\xi^x)\varepsilon,$$

(4. 7) $$\delta g_{cb}=-2h_{cbx}\xi^x\varepsilon.$$

Thus we have from (3. 9), (4. 5), (4. 6) and (4. 7)

THEOREM 4. 1. *Suppose that the variation of f-invariant and k-invariant submanifold with induced (f, g, u, v, λ)-structure of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is normal. Then the variation is isometric if and only if it is f-preserving.*

From the first equation of (4. 5), we have

PROPOSITION 4. 2. *An f-invariant and k-invariant submanifold with induced (f, g, u, v, λ)-structure of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ is minimal.*

Furthermore we have from (3. 9)

(4. 9) $$\delta(f_b{}^a f_a{}^c)=(\delta u_b)u^c+u_b(\delta u^c)+(\delta v_b)v^c+v_b(\delta v^c).$$

If the variation preserves $f_b{}^a$ and u^a, we have from (4. 9)

(4. 10) $$(\delta u_b)u^c+(\delta v_b)v^c+v_b(\delta v^c)=0.$$

Transvecting (4. 10) with u_c and v_c, we find respectively

(4. 11) $$(1-\lambda^2)\delta u_b+u_c v_b(\delta v^c)=0,$$

$$(4.12)\qquad (1-\lambda^2)\delta v_b+v_c v_b(\delta v^c)=0.$$

Transvecting (4.11) with u^b, $(\delta u_b)u^b=0$. Then

$$\delta(u^b u_b)=-2\lambda(\delta\lambda)=0,\quad \text{that is,}\quad \delta\lambda=0.$$

Applying the operator δ to (3.11), we can get $\delta v^a=0$ from above. So from (4.11) and (4.12), $\delta u_b=0$ and $\delta v_b=0$. Thus we have

PROPOSITION 4.3. *If an infinitesimal f-preserving variation of the submanifold with induced (f, g, u, v, λ)-structure preserves u^a, then the variation preserves u_a, v^a, v_a and λ.*

Finally, when the submanifold with induced (f, g, u, v, λ)-structure is f-invariant and k-invariant, we get from the Ricci-identity, (1.17), (2.9), (2.16), (3.2) and (3.20)

$$(4.13)\qquad K_{dce}{}^a f_b{}^e-K_{dcb}{}^e f_e{}^a=-g_{cb}f_d{}^a+g_{db}f_c{}^a+\delta_c^a f_{db}-\delta_d^a f_{cb}$$
$$+k_d{}^e(f_{eb}k_c{}^a-k_{cb}f_e{}^a)+k_c{}^e(f_e{}^a k_{db}-k_d{}^a f_{eb}).$$

Transvecting (4.13) with $f_a{}^d$ and using (3.9), (3.22) and (1.17), we get

$$(4.14)\qquad K_{dcea}f_b{}^e f^{ad}=-K_{cb}+(u_e u^d+v_e v^d)K_{dcb}{}^e+(m-4+2\lambda^2)g_{cb}$$
$$+2(u_c u_b+v_c v_b)+K_d{}^d K_{cb}.$$

Transvecting (1.17) with $f_c{}^d$, we have

$$(4.15)\qquad K_d{}^d=0.$$

From (4.14) and (4.15), we find

$$(4.16)\qquad K_{dcea}f_b{}^e f^{ad}=-K_{cb}+(u_e u^d+v_e v^d)K_{dcb}{}^e$$
$$+(m-4+2\lambda^2)g_{cb}+2(u_c u_b+v_c v_b).$$

Differentiating λ covariantly along M^m and using the last equation of (2.8), we get

$$(4.17)\qquad \nabla_c\lambda=-2v_c.$$

Using the Ricci-identity, (2.9), (2.16), (3.2), (3.20), (3.22) and (4.17), we obtain

$$(4.18)\qquad K_{dcbe}(u^e u^d+v^e v^d)=2(1-\lambda^2)g_{cb}-2(u_c u_b+v_c v_b).$$

Thus we can get the following useful identity from (4.16) and (4.18) for later use

$$(4.19)\qquad K_{dcea}f_b{}^e f^{ad}=-K_{cb}+(m-2)g_{cb}.$$

5. Some integral formulas.

In this section, on a compact f-invariant and k-invariant submanifold with induced (f, g, u, v, λ)-structure, we investigate some variation-preserving relations. Through this section, we assume that the submanifold M^m of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ has induced (f, g, u, v, λ)-structure and is f-invariant and k-invariant.

First of all, we define T_{ba} by

$$(5.1)\qquad T_{ba}=(\nabla_b\xi^e-h_b{}^e{}_x\xi^x)f_{ea}-f_b{}^e(\nabla_e\xi_a-h_{eax}\xi^x)$$
$$-\xi_b u_a+u_b\xi_a-\xi^e k_{eb}v_a+v_b\xi^e k_{ea}.$$

Then we find that a variation of the submanifold preserves $f_b{}^a$ if and only if $T_{cb}=0$.

If we take account of (3.9), (3.11), (3.22), (2.9), (2.16) and (3.18), we have

$$(5.2)\qquad T_{ba}T^{ba}=2(\nabla_b\xi_a)(\nabla^b\xi^a)-8h^{cb}{}_x\xi^x(\nabla_c\xi_b)+4(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y)$$
$$-(u^e u_a+v^e v_a)[(\nabla_b\xi_e)(\nabla^b\xi^a)+(\nabla_e\xi_b)(\nabla^a\xi^b)]$$
$$-2\lambda v_a\xi_b[(\nabla^b\xi^a)-(\nabla^a\xi^b)]+2u_b\xi_a f_c{}^a(\nabla^b\xi^c-\nabla^c\xi^b)$$
$$-2f_b{}^e f_c{}^a(\nabla_e\xi_a)(\nabla^b\xi^c)$$
$$+2\lambda u_c\xi^e k_{eb}(\nabla^b\xi^c-\nabla^c\xi^b)+2v_b\xi^e k_{ea}f_c{}^a(\nabla^b\xi^c-\nabla^c\xi^b)$$
$$+4(1-\lambda^2)\xi^a\xi_a-4(u^a\xi_a)^2+4(v^a\xi_a)^2.$$

From (3.2), (3.22) and (4.15), we get

$$(5.3)\qquad \nabla_b f^{be}=-mu^e.$$

On the other hand, we have

$$(5.4)\qquad \nabla_b W^b=(\nabla_b\nabla^b\xi^c)\xi_c+(\nabla^b\xi^c)(\nabla_b\xi_c)$$
$$+mu^e f^{ac}\xi_a(\nabla_e\xi_c)+u^e f^{ac}\xi_a(\nabla_c\xi_e)$$
$$-u^a\xi_a f^{ce}(\nabla_e\xi_c)$$
$$+f^{be}k_b{}^c(\xi_c v^a\nabla_e\xi_a-v^a\xi_a\nabla_e\xi_c)$$
$$-f^{be}f^{ac}(\nabla_b\nabla_e\xi_c)\xi_a-f^{be}f^{ac}(\nabla_e\xi_c)(\nabla_b\xi_a)$$

because of (3.2) and (4.15), where we have put

$$W^b=(\nabla^b\xi^c)\xi_c-f^{be}f^{ac}(\nabla_e\xi_c)\xi_a,$$

from which, using the Ricci-identity and (4.19),

$$(5.5)\qquad \begin{aligned}\nabla_b W^b=&(\nabla_b\nabla^b\xi^c)\xi_c+(\nabla^b\xi^c)(\nabla_b\xi_c)\\&+mu^ef^{ac}\xi_a(\nabla_e\xi_c)+u^ef^{ac}\xi_a(\nabla_c\xi_e)\\&-f^{ce}u^a\xi_a(\nabla_e\xi_c)\\&+f^{be}k_b{}^c(\xi_c v^a\nabla_e\xi_a-v^a\xi_a\nabla_e\xi_c)\\&+K_{cb}\xi^c\xi^b-(m-2)\xi_c\xi^c-f^{be}f^{ac}(\nabla_e\xi_c)(\nabla_b\xi_a).\end{aligned}$$

Comparing (5.2) with (5.5), we have

$$(5.6)\qquad \begin{aligned}T_{ba}T^{ba}=&2(\nabla_b W^b)-2\xi^c(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)\\&+8h^{cb}{}_x\xi^x(\nabla_c\xi_b)+4(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y)\\&-(u^eu_a+v^ev_a)[(\nabla_b\xi_e)(\nabla^b\xi^a)+(\nabla_e\xi_b)(\nabla^a\xi^b)]\\&-2\lambda v_a\xi_b[(\nabla^b\xi^a-\nabla^a\xi^b)]+2\lambda u_c\xi^ek_{eb}(\nabla^b\xi^c-\nabla^c\xi^b)\\&-2(m+1)u^ef^{ac}\xi_a(\nabla_e\xi_c)+2v_b\xi^ek_{ea}f_c{}^a\nabla^b\xi^c\\&+2f^{ce}u^a\xi_a(\nabla_e\xi_c)+2f^{be}k_b{}^cv^a\xi_a(\nabla_e\xi^c)\\&+2(m-2\lambda^2)\xi^a\xi_a-4[(\xi^bu_b)^2-(\xi^bv_b)^2],\end{aligned}$$

or equivalently

$$(5.7)\qquad \begin{aligned}T^{ba}T_{ba}=&2\nabla_b(W^b-2h^{cb}{}_x\xi_c\xi^x)\\&-2\xi^c[\nabla^b\nabla_b\xi_c+K_{cb}\xi^b-2\nabla^b(h_{cbx}\xi^x)]\\&-2h^{cb}{}_y\xi^y(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\\&-(u^eu_a+v^ev_a)[(\nabla_b\xi_e)(\nabla^b\xi^a)+(\nabla_e\xi_b)(\nabla^a\xi^b)]\\&-2\lambda v_a\xi_b(\nabla^b\xi^a-\nabla^a\xi^b)+2\lambda u_c\xi^ek_{eb}(\nabla^b\xi^c-\nabla^c\xi^b)\\&-2(m+1)f^{ac}\xi_au^e(\nabla_e\xi_c)+2v_b\xi^ek_{ea}f_c{}^a(\nabla^b\xi^c)\\&+2f^{ce}(\nabla_e\xi_c)u^a\xi_a+2v^a\xi_af^{be}k_b{}^c(\nabla_e\xi_c)\\&+2(m-2\lambda^2)\xi^a\xi_a-4[(u^b\xi_b)^2-(v^b\xi_b)^2].\end{aligned}$$

Thus we assume that the submanifold M^m is compact. Using

$$\begin{aligned}\nabla_e(u^a\xi_a\xi_cf^{ce})=&-\xi_c\xi^c+(m+1)(u^a\xi_a)^2+(v^a\xi_a)^2\\&+u^a\xi_af^{ce}(\nabla_e\xi_c)-\lambda\xi_c\xi_ak_e{}^af^{ce}+u^a\xi_cf^{ce}(\nabla_e\xi_a)\end{aligned}$$

and

$$\begin{aligned}\nabla_e(v^a\xi_a\xi_cf^{be}k_b{}^c)=&\xi_c\xi^c-(u^a\xi_a)^2-(m+1)(v^a\xi_a)^2\\&+\lambda\xi_c\xi_af^{ba}k_b{}^c+v^a\xi_cf^{be}k_b{}^c(\nabla_e\xi_a)\end{aligned}$$

$$+v^a\xi_a f^{be}k_b{}^c(\nabla_e\xi_c)$$

and (5.7), we apply Green's theorem and obtain

$$(5.8)\qquad \int_{M^m}[T^{ba}T_{ba}+2\xi^c\{\nabla^b\nabla_b\xi_c+K_{cb}\xi^b-2\nabla^b(h_{cbx}\xi^x)\}$$
$$+2h^{cb}{}_y\xi^y(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)$$
$$+(u^eu_a+v^ev_a)\{(\nabla_b\xi_e)(\nabla^b\xi^a)+(\nabla_e\xi_b)(\nabla^a\xi^b)\}$$
$$+2\lambda v_a\xi_b(\nabla^b\xi^a-\nabla^a\xi^b)-2\lambda u_c\xi^e k_{eb}(\nabla^b\xi^c-\nabla^c\xi^b)$$
$$+2(m+1)f^{ac}\xi_a u^e(\nabla_e\xi_c)+2f^{ce}\xi_c u^a(\nabla_e\xi_a)$$
$$+2v_b\xi^e k_e{}^a f_{ac}(\nabla^b\xi^c+\nabla^c\xi^b)+4\lambda\xi_c\xi^a k_{ae}f^{ec}$$
$$-2(m-2\lambda^2)\xi^a\xi_a+2(m+2)\{(u^b\xi_b)^2-(v^b\xi_b)^2\}]dV=0,$$

dV being the volume element of M^m.

From (4.1) and (4.2), the variation of dV is given by [2]

$$(5.9)\qquad \delta dV=(\nabla_a\xi^a-h_a{}^a{}_x\xi^x)dV\varepsilon.$$

For a compact orientable submanifold M^m, we have the following integral formula:

$$\int_{M^m}[\xi^c(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)$$
$$+\frac{1}{2}(\nabla_c\xi_b+\nabla_b\xi_c)(\nabla^c\xi^b+\nabla^b\xi^c)-(\nabla_b\xi^b)^2]dV=0,$$

which is valid for any vector ξ^c in M^m [4], from which

$$(5.10)\qquad \int_{M^m}[\xi^c\{(\nabla^b\nabla_b\xi_c+K_{cb}\xi^b)-2\nabla^b(h_{cb}{}^x\xi_x)+\nabla_c(h_b{}^b{}_x\xi^x)\}$$
$$+\frac{1}{2}(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cby}\xi^y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x)$$
$$-(\nabla_c\xi^c-h_c{}^c{}_x\xi^x)(\nabla_b\xi^b)$$
$$+(h^{cb}{}_y\xi^y)(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)]dV=0.$$

Thus we have an integral formula from proposition (4.2), (5.8) and (5.10)

$$(5.11)\qquad \int_{M^m}[T^{ba}T_{ba}+(u^eu_a+v^ev_a)\{(\nabla_b\xi_e)(\nabla^b\xi^a)+(\nabla_e\xi_b)(\nabla^a\xi^b)\}$$
$$+2\lambda v_a\xi_b(\nabla^b\xi^a-\nabla^a\xi^b)-2\lambda u_c\xi^e k_{eb}(\nabla^b\xi^c-\nabla^c\xi^b)$$
$$+2(m+1)f^{ac}\xi_a u^e(\nabla_e\xi_c)+2f^{ce}\xi_c u^a(\nabla_e\xi_a)$$

$$+2v_b\xi^e k_e{}^a f_{ac}(\nabla^b\xi^c+\nabla^c\xi^b)+4\lambda k_{ae}f^{ec}\xi_c\xi^a$$
$$-2(m-2\lambda^2)\xi^a\xi_a+2(m+2)\{(u^b\xi_b)^2-(v^b\xi_b)^2\}$$
$$-(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cby}\xi^y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{bc}{}_x\xi^x)$$
$$+2(\nabla_c\xi^c)^2]dV=0.$$

Now we assume that the variation of the submanifolds preserves u^a, v^a, u_a and v_a. Then we have from (2.12), (2.17) and (4.1)

$$u_b\nabla^b\xi^a=f_b{}^a\xi^b-\lambda\xi^b k_b{}^a,\quad v_b\nabla^b\xi^a=\xi^c k_c{}^b f_b{}^a+\lambda\xi^a, \tag{5.12}$$

$$u^a\nabla_c\xi_a=\lambda\xi^b k_{bc}-f_{bc}\xi^b,\quad v^a\nabla_c\xi_a=-\lambda\xi_c-\xi^d k_d{}^b f_{bc}, \tag{5.13}$$

from which, we get

$$u^b\nabla_a\xi_b=-u^b\nabla_b\xi_a,\quad v^b\nabla_a\xi_b=-v^b\nabla_b\xi_a. \tag{5.14}$$

Using (5.11)～(5.14), we get

$$\int_{M^m}[T^{ba}T_{ba}+2(m+2)\lambda\xi^e\xi_a k_e{}^b f_b{}^a \tag{5.15}$$
$$+4\xi^a\xi_a-4(m+2)(v^b\xi_b)^2+2(\nabla_c\xi^c)^2$$
$$-(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cb}{}^y\xi_y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x)]dV=0.$$

To obtain the variation of λ, that is, $\delta\lambda=\lambda(x+\xi\varepsilon)-\lambda(x)$, using the last equation of (2.8), we have

$$\delta\lambda=-2\xi^a v_a\varepsilon. \tag{5.16}$$

By hypothesis, proposition (4.3) and (5.16), we find

$$v^a\xi_a=0. \tag{5.17}$$

From (5.13), we have $\lambda\xi_a=-v^b\nabla_a\xi_b-\xi^d k_d{}^d f_{ba}$. Using the above equation and (5.17), we have

$$\lambda\xi_a(\xi_c k^{ce}f_e{}^a)=-\xi^d\xi_d+(\xi^a u_a)^2+\xi_c k^{ce}f_e{}^a v^b(\nabla_b\xi_a), \tag{5.18}$$

$$\lambda\xi_c(\xi_a k^{ce}f_e{}^a)=-\xi^d\xi_d+(\xi^a u_a)^2+\xi_a k^{ce}f_e{}^a v^b(\nabla_b\xi_c). \tag{5.19}$$

We get from (5.18) and (5.19)

$$\xi_c k^{ce}f_e{}^a v^b(\nabla_b\xi_a)=\xi_a k^{ce}f_e{}^a v^b(\nabla_b\xi_c). \tag{5.20}$$

Using $\nabla_b(v^b\xi_a\xi_c k^{ce}f_e{}^a)=m\lambda\xi_a\xi_c k^{ce}f_e{}^a+2(u^c\xi_c)^2-2(v^c\xi_c)^2$
$$+v^b(\nabla_b\xi_a)\xi_c k^{ce}f_e{}^a+v^b(\nabla_b\xi_c)\xi_a k^{ce}f_e{}^a$$

and (5.18), we apply Green's theorem and obtain

$$(5.21)\quad \int_{M^m}(\lambda\xi_a\xi_c k^{ce}f_e{}^a)\,dV=\int_{M^m}[-\xi^d\xi_d-(u^a\xi_a)^2 \\ -m\lambda\xi_a\xi_c k^{ce}f_e{}^a-v^b(\nabla_b\xi_c)\xi_a k^{ce}f_e{}^a]\,dV.$$

From (5.19) and (5.21), we find

$$(5.22)\quad \int_{M^m}(\lambda\xi_a\xi_c k^{ce}f_e{}^a)\,dV=\frac{-2}{m+2}\int(\xi^a\xi_a)\,dV.$$

Thus we have an integral formula from (5.15), (5.17) and (5.22)

$$(5.23)\quad \int_{M^m}[T^{ba}T_{ba}-(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cb}{}^y\xi_y)(\nabla^c\xi^b+\nabla^b\xi^c-2h^{cb}{}_x\xi^x) \\ +2(\nabla_c\xi^c)^2dV=0.$$

From this integral formula, proposition 4.2 and (5.9), we have

PROPOSITION 5.1. *Suppose that a variation of the f–invariant and k–invariant compact submanifold with induced (f, g, u, v, λ)–structure of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ preserves u^a, u_a, v^a and v_a. Then in order for the variation to be isometric it is necessary and sufficient that the variation preserves volume and $f_b{}^a$.*

Furthermore, if the variation of the submanifold is affine, we have from (4.4)

$$\nabla_c\nabla_b\xi_a+K_{dcba}\xi^d-\nabla_c(h_{bax}\xi^x)-\nabla_b(h_{cax}\xi^x)+\nabla_a(h_{cbx}\xi^x)=0,$$

from which, using Proposition 4.2 $\nabla_c(\nabla_a\xi^a)=0$, that is, $\nabla_a\xi^a=$constant. Thus assuming the submanifold to be compact, we have $\nabla_a\xi^a=0$. From this fact and proposition 5.1, we obtain

THEOREM 5.2. *Assume that the variation of a compact f–invariant and k–invariant submanifold with induced (f, g, u, v, λ)–structure of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ preserves u^a, u_a, v^a and v_a. Then the variation is isometric if and only if it is affine and preserves $f_b{}^a$.*

Moreover, we have from proposition 4.3

COROLLARY 5.3. *Suppose that the variation of a compact f–invariant and k–invariant submanifold with induced (f, g, u, v, λ)–structure of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ preserves u^a and $f_b{}^a$. Then the variation is affine if and only if it is isometric.*

6. Infinitesimal variations of *f*-antiinvariant submanifold and *k*-antiinvariant one.

We assume that M^m is an f–antiinvariant submanifold of $S^n(1/\sqrt{2})\times S^n$

$(1/\sqrt{2})$.
We then have

$$f_i{}^h B_b{}^i = -f_b{}^x C_x{}^h,\quad f_i{}^h C_y{}^i = f_y{}^a B_a{}^h + f_y{}^x C_x{}^h. \tag{6.1}$$

Differentiating the first equation of (6.1) covariantly along M^m, using (1.8) and (1.13), we find

$$\begin{aligned}&[-g_{cb}u^a + u_b\delta_c{}^a - k_{cb}v^a + v_b k_c{}^a + h_{cb}{}^x f_x{}^a]B_a{}^h\\&+[-g_{cb}u^x - k_{cb}v^x + v_b k_c{}^x + h_{cb}{}^y f_y{}^x]C_x{}^h\\&= f_b{}^x h_c{}^a{}_x B_a{}^h - (\nabla_c f_b{}^x)C_x{}^h,\end{aligned}$$

and consequently, comparing the tangential and normal parts

$$h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x - g_{cb}u^a + \delta_c{}^a u^b - k_{cb}v^a + v_b k_c{}^a = 0 \tag{6.2}$$

and

$$\nabla_c f_b{}^x = g_{cb}u^x - h_{cb}{}^y f_y{}^x + k_{cb}v^x - v_b k_c{}^x. \tag{6.3}$$

From (6.2), taking the skew symmetric part, we have

$$h_c{}^a{}_x f_b{}^x - h_b{}^a{}_x f_c{}^x = \delta_c{}^a u_b - \delta_b{}^a u_c + v_b k_c{}^a - v_c k_b{}^a. \tag{6.4}$$

Differentiating the second equation of (6.1) covariantly along M^m, using (1.8) and (1.13), we find

$$\begin{aligned}&(u_y\delta_c{}^a - k_{cy}v^a + v_y k_c{}^a)B_a{}^h + (-k_{cy}v^x + v_y k_c{}^x + h_c{}^a{}_y f_a{}^x)C_x{}^h\\&= (-h_c{}^a{}_x f_y{}^x + \nabla_c f_y{}^a)B_a{}^h + (f_y{}^a h_{ca}{}^x + \nabla_c f_y{}^x)C_x{}^h.\end{aligned}$$

Thus, comparing the tangential and normal parts, we have

$$\nabla_c f_y{}^a = \delta_c{}^a u_y + h_c{}^a{}_x f_y{}^x - k_{cy}v^a + v_y k_c{}^a, \tag{6.5}$$

which is equivalent to (6.3) and

$$\nabla_c f_y{}^x = h_c{}^a{}_y f_a{}^x - h_{ce}{}^x f_y{}^e - k_{cy}v^x + v_y k_c{}^x. \tag{6.6}$$

We now consider an infinitesimal variation (2.1) and assume that it carries the f-antiinvariant submanifold M^m into a f-antiinvariant submanifold. Then we have

$$f_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i = -(f_b{}^x + \delta f_b{}^x)\bar{C}_x{}^h,$$

that is, using (2.6) and (2.7),

$$\begin{aligned}&(f_i{}^h + \xi^j\partial_j f_i{}^h\varepsilon)(B_b{}^i + \partial_b\xi^i\varepsilon)\\&= -(f_b{}^x + \delta f_b{}^x)[C_x{}^h - \Gamma_{ji}{}^h\xi^j C_x{}^i\varepsilon + (\eta_x{}^a B_a{}^h + \eta_x{}^y C_y{}^h)\varepsilon],\end{aligned}$$

from which, using (1.8), we obtain

$$[f_i^{\ h}+\xi^j(-\Gamma_{jt}^{\ \ h}f_i^{\ t}+\Gamma^t{}_{ji}f_t^{\ h}-g_{ji}u^h+\delta_j^{\ h}u_i$$
$$-k_{ji}v^h+k_j^{\ h}v_i)\varepsilon](B_b^{\ i}+\partial_b\xi^i\varepsilon)$$
$$=-(f_b^{\ y}+\delta f_b^{\ y})[C_y^{\ h}-\Gamma_{ji}^{\ \ h}\xi^jC_y^{\ i}\varepsilon+(\eta_y^{\ a}B_a^{\ h}+\eta_y^{\ x}C_x^{\ h})\varepsilon],$$

that is,

$$[f_i^{\ h}(\nabla_b\xi^i)-\xi_bu^h+u_b\xi^h]\varepsilon$$
$$=-f_b^{\ y}(\eta_y^{\ a}B_a^{\ h}+\eta_y^{\ x}C_x^{\ h})\varepsilon-(\delta f_b^{\ y})C_y^{\ h}$$
$$+[(\xi_ck_b^{\ c}+\xi_yk_b^{\ y})v^a-v_b(\xi^ck_c^{\ a}+\xi^xk_x^{\ a})]B_a^{\ h}\varepsilon$$
$$+[(\xi_ck_b^{\ c}+\xi_yk_b^{\ y})v^x-v_b(\xi^ak_a^{\ x}+\xi^yk_y^{\ x})]C_x^{\ h}\varepsilon.$$

Thus substituting (2.4) and (6.1) into the above equation, we find

$$[(\nabla_b\xi^x+h_{be}{}^x\xi^e)f_x^{\ a}-\xi_bu^a+u_b\xi^a]B_a^{\ h}\varepsilon$$
$$+[-(\nabla_b\xi^a-h_b{}^a{}_x\xi^x)f_a^{\ y}+(\nabla_b\xi^x+h_{ba}{}^x\xi^a)f_x^{\ y}-\xi_bu^y+u_b\xi^y]C_y^{\ h}\varepsilon$$
$$=[-f_b^{\ y}\eta_y^{\ a}+(\xi_ck_b^{\ c}+\xi_yk_b^{\ y})v^a-v_b(\xi^ck_c^{\ a}+\xi^xk_x^{\ a})]B_a^{\ h}\varepsilon-(\delta f_b^{\ y})C_y^{\ h}$$
$$+[-f_b^{\ x}\eta_x^{\ y}+(\xi_ck_b^{\ c}+\xi_xk_b^{\ x})v^y-v_b(\xi^ak_a^{\ y}+\xi^xk_x^{\ y})]C_y^{\ h}\varepsilon,$$

from which, comparing tangential and normal parts and using (2.8),

$$(6.7)\qquad (\nabla_b\xi^y+h_{be}{}^y\xi^e)f_y^{\ a}-\xi_bu^a+u_b\xi^a$$
$$=f_b^{\ y}(\nabla^a\xi_y+h_e{}^a{}_y\xi^e)+(\xi_ck_b^{\ c}+\xi_yk_b^{\ y})v^a-v_b(\xi^ek_e^{\ a}+\xi^yk_y^{\ a}),$$

which is, according to (6.2), equivalent to

$$(6.8)\qquad (\nabla_b\xi^y)f_y^{\ a}=f_b^{\ y}(\nabla^a\xi_y)+v^x\xi_yk_b^{\ y}-v_b\xi^yk_y^{\ a},$$

and

$$(6.9)\qquad \delta f_b^{\ x}=[(\nabla_b\xi^a-h_b{}^a{}_y\xi^y)f_a^{\ x}-(\nabla_b\xi^y+h_{be}{}^y\xi^e)f_y^{\ x}$$
$$+\xi_bu^x-u_b\xi^x-f_b^{\ y}\eta_y^{\ x}$$
$$+(\xi_ck_b^{\ c}+\xi_yk_b^{\ y})v^x-v_b(\xi^ak_a^{\ x}+\xi^yk_y^{\ x})]\varepsilon.$$

Thus we have

THEOREM 6.1. *In order for an infinitesimal variation* (2.1) *to carry an f-antiinvariant submanifold* M^m *of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *into f-antiinvariant one, it is necessary and sufficient that* (6.8) *holds, the variation of* $f_b^{\ x}$ *being given by* (6.9).

Now we assume that M^m is a k-antiinvariant submanifold of $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$. We then have

$$(6.10)\qquad k_i^{\ h}B_b^{\ i}=k_b^{\ x}C_x^{\ h},\quad k_i^{\ h}C_y^{\ i}=k_y^{\ a}B_a^{\ h}+k_y^{\ x}C_x^{\ h}.$$

Differentiating the first equation of (6. 10) covariantly along M^m and using (1. 8), we get

$$h_{cb}{}^x k_x{}^a = -h_c{}^a{}_x k_b{}^x, \tag{6.11}$$

$$\nabla_c k_b{}^x = h_{cb}{}^y k_y{}^x. \tag{6.12}$$

From (6. 11), we find

$$h_{cb}{}^x k_x{}^a = 0. \tag{6.13}$$

Differentiating the second equation of (6. 1) covariantly along M^m and using (1. 8), we have

$$\nabla_c k_y{}^a = h_{cx}{}^a k_y{}^x, \tag{6.14}$$

which is equivalent to (6. 12) and

$$\nabla_c k_y{}^x = -k_y{}^a h_{ca}{}^x - h_c{}^a{}_y k_a{}^x. \tag{6.15}$$

We now consider an infinitesimal variation (2. 1) and assume that it carries a k-antiinvariant submanifold M^m into a k-antiinvariant one. Then we have

$$k_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i = (k_b{}^x + \delta k_b{}^x)\bar{C}_x{}^h),$$

that is, using (2. 6) and (2. 7),

$$\begin{aligned}&(k_i{}^h + \xi^j \partial_j k_i{}^h \varepsilon)(B_b{}^i + \partial_b \xi^i \varepsilon)\\ &= (k_b{}^x + \delta k_b{}^x)[C_x{}^h - \Gamma_{ji}{}^h \xi^j C_x{}^i \varepsilon + (\eta_x{}^a B_a{}^h + \eta_x{}^y C_y{}^h)\varepsilon],\end{aligned}$$

from which, using (1. 8), we obtain

$$\begin{aligned}&[k_i{}^h + \xi^j(-\Gamma_{jt}{}^h k_i{}^t + \Gamma_{ji}{}^t k_t{}^h)\varepsilon](B_b{}^i + \partial_b \xi^i \varepsilon)\\ &= (k_b{}^x + \delta k_b{}^x)[C_x{}^h - \Gamma_{ji}{}^h \xi^j C_x{}^i \varepsilon + (\eta_x{}^a B_a{}^h + \eta_x{}^y C_y{}^h)\varepsilon],\end{aligned}$$

that is,

$$k_i{}^h(\nabla_b \xi^i)\varepsilon = k_b{}^x(\eta_x{}^a B_a{}^h + \eta_x{}^y C_y{}^h)\varepsilon + (\delta k_b{}^x)C_x{}^h.$$

Thus substituting (2. 4) and (6. 10) into the above equation,

$$\begin{aligned}(\delta k_b{}^x)C_x{}^h = &[-k_b{}^x \eta_x{}^a + (\nabla_b \xi^y)k_y{}^a + \xi^c h_{bc}{}^y k_y{}^a]B_a{}^h \varepsilon\\ &+ [(\nabla_b \xi^c - h_b{}^c{}_y \xi^y)k_c{}^x + (\nabla_b \xi^y + h_{bc}{}^y \xi^c)k_y{}^x - k_b{}^y \eta_y{}^x]C_x{}^h \varepsilon,\end{aligned}$$

from which, comparing tangential and normal parts, using (2. 8) and (6. 13),

$$k_b{}^x(\nabla^a \xi_x) + (\nabla_b \xi^y)k_y{}^a = 0, \tag{6.16}$$

$$\delta k_b{}^x = [-k_b{}^y \eta_y{}^x + (\nabla_b \xi^c - h_b{}^c{}_y \xi^y)k_c{}^x \tag{6.17}$$

$$+(\nabla_b\xi^y+h_{bc}{}^y\xi^c)k_y{}^x]\varepsilon.$$

Thus we have

THEOREM 6.2. *In order for an infinitesimal variation* (2.1) *to carry a k-antiinvariant submanifold* M^m *of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *into a k-antiinvariant one, it is necessary and sufficient that* (6.16) *holds, the variation of* $f_b{}^x$ *being given by* (6.17).

An infinitesimal variation given by (2.1) is called a *k-antiinvariant variation* if it carries a k-antiinvariant submanifold into a k-antiinvariant one.

Now from (2.5), (6.13) and (6.16), we have

THEOREM 6.3. *A parallel variation of a k-antiinvariant submanifold of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *is k-antiinvariant variation.*

On the other hand, from (1.9), (6.10) and (6.11) we find

$$k_b{}^x k_x{}^a=\delta_b^a,\quad k_b{}^x k_x{}^y=0,\quad k_y{}^a k_a{}^x+k_y{}^z k_z{}^x=\delta_y^x. \tag{6.18}$$

Assume that the variation of k-antiinvariant submanifold M^m preserves $k_b{}^x$, we get from (6.17)

$$-k_b{}^y\eta_y{}^x+(\nabla_b\xi^c-h_b{}^c{}_y\xi^y)k_c{}^x+(\nabla_b\xi^y+h_{bc}{}^y\xi^c)k_y{}^x=0. \tag{6.19}$$

Then (6.18) and (6.19) imply

$$\nabla_b\xi_a-h_{bay}\xi^y=k_b{}^y k_a{}^x\eta_{yx}. \tag{6.20}$$

Thus, by (4.1), (6.20) and $\eta_{yx}=-\eta_{xy}$, $\delta g_{cb}=0$. Therefore we have

THEOREM 6.4. *If the variation of k-antiinvariant submanifold of* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *preserves* $k_b{}^x$, *then the variation is isometric.*

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