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# MEROMORPHIC CLOSE-TO-STAR FUNCTIONS

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## 1. Introduction

In [4], Reade introduced the class of close-to-star functions. A function $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, analytic in the unit disk $E(|z|<1)$, is said to be close-to-star if there exists a starlike function $g(z)=z+\sum_{n=2}^{\infty} b_n z^n$ for $z$ in $E$ such that the condition

$$\text{(1.1)} \qquad \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\}>0$$

holds for all $z$ in $E$.

Let $E_0$ be the set of complex numbers $z$ for which $0<|z|<1$. A function $g(z)=1/z+\sum_{n=0}^{\infty} b_n z^n$, analytic in $E_0$, is called meromorphic starlike function of order $r$, $g\in\Sigma^*(r)$, if

$$\text{(1.2)} \qquad \operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\}<r \quad (z\in E)$$

holds for some $r(0\leq r<1)$. These functions and some of its consequences are discussed in [3].

Motivated by Reade [4], Padmanabhan [2] has defined the class '$\Gamma(\alpha)$' of meromorphic close-to-star functions of order $\alpha(0\leq\alpha<1)$. A function $f(z)=1/z+\sum_{n=0}^{\infty} a_n z^n$, analytic in $E_0$, is in the class $\Gamma(\alpha)$ if there exists a meromorphic starlike function $g\in\Sigma^*(0)$ such that

$$\text{(1.3)} \qquad \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\}>\alpha$$

holds for some $\alpha(0\leq\alpha<1)$ and for all $z$ in $E$. Here we introduce the concept of 'type' in the class of meromorphic close-to-star functions of order $\alpha$.

A function

$$\text{(1.4)} \qquad f(z)=1/z+a_0+a_1z+\cdots+a_nz^n+\cdots$$

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analytic in the punctured disk $E_0=\{z : 0<|z|<1\}$, is said to be a meromorphic close-to-star of order $\alpha$ and type $\beta, f\in\Gamma(\alpha,\beta)$, if there exists a meromorphic starlike function $g\in\Sigma^*(0)$ such that the condition

$$(1.5)\quad |(f(z)/g(z)-1)/\{2\beta(f(z)/g(z)-\alpha)-(f(z)/g(z)-1)\}|<1$$

holds for some $\alpha$, $\beta(0\leq\alpha<1,\ 0<\beta\leq1)$ and for all $z$ in $E$. Denote by $\Gamma^r(\alpha,\beta)$, $0\leq\alpha<1$, $0<\beta\leq1$, $0\leq r<1$, the family of functions $f$ given by (1.4) which are analytic in $E_0$ together with some $g\in\Sigma^*(r)$ satisfy the condition (1.5) for all $z$ in $E$. Evidently $\Gamma^0(\alpha,\beta)\equiv\Gamma(\alpha,\beta)$ and $\Gamma^0(\alpha,1)\equiv\Gamma(\alpha)$.

For some suitable values of the parameters $\alpha$ and $\beta$, our class $\Gamma^r(\alpha,\beta)$ gives rise to many new subclasses of meromorphic close-to-star functions; for example

$$(1.6)\quad \Gamma^r{}_{\alpha,\beta}=\{f\in N^* : |(f(z)/g(z)-1)/(f(z)/g(z)+1-2\alpha)| <\beta,\ g\in\Sigma^*(r),\ 0\leq\alpha<1,\ 0<\beta\leq1,\ z\in E\},$$

$$(1.7)\quad \Lambda^r(\alpha)=\{f\in N^* : |f(z)/g(z)-1/(2\alpha)|<1/(2\alpha),\ g\in\Sigma^*(r),\ 0\leq\alpha<1,\ z\in E\},$$

$$(1.8)\quad G^r(\delta)=\{f\in N^* : |f(z)/g(z)-\delta|<\delta,\ g\in\Sigma^*(r),\ \delta>1/2,\ z\in E\},$$

where $N^*$ denotes the class of functions $f$ of the form (1.4) that are analytic in the punctured disk $E_0$. We remark that $\Gamma^r{}_{\alpha,\beta}\equiv\Gamma^r((1-\beta+2\alpha\beta)/(1+\beta),\ (1+\beta)/2)$, $\Lambda^r(\alpha)\equiv\Gamma^r(0,1-\alpha)$ and $G^r(\delta)\equiv\Gamma^r(0,(2\delta-1)/(2\delta))$.

In this paper, we are interested mainly in the determination of the radii of starlikeness for the functions in the class $\Gamma^r(\alpha,\beta)$.

## 2. Radii of starlikeness of $\Gamma^r(\alpha,\beta)$

Let $B$ denote the class of analytic functions $\omega$ in $E$ which satisfy the conditions (i) $\omega(0)=0$ and (ii) $|\omega(z)|<1$ for $z$ in $E$.

THEOREM 1. *Let $f\in\Gamma^r(\alpha,\beta)$, with $g\in\Sigma^*(r)$. Then for $|z|=r$, $0<r<1$, we have*

$$(2.1)\quad Re\left\{-\frac{zf'(z)}{f(z)}\right\}\geq\begin{cases}\dfrac{1+lr}{1+r}+\dfrac{s+t}{s-t}-\dfrac{2(1-str^2)}{(s-t)(1-r^2)}\\ \qquad+\dfrac{2\sqrt{(1-s)(1-t)(1+sr^2)(1+tr^2)}}{(s-t)(1-r^2)}, & R_1\leq R_2\\ \dfrac{1+lr}{1+r}-\dfrac{(s-t)r}{(1-sr)(1-tr)}, & R_1\geq R_2\end{cases}$$

*where $s=2\beta-1$, $t=2\alpha\beta-1$, $l=2r-1$, $-1\leq t<s<1$, $-1\leq l<1$ and*

$$R_1{}^2=\frac{(1-t)(1+tr^2)}{(1-s)(1+sr^2)},\quad R_2=\frac{1-tr}{1-sr}.$$

*Furthermore, for* $s=1$, $-1\leq t<1$, $-1\leq l<1$ *and* $|z|=r$, $0<r<1$, *we have*

$$(2.2)\qquad Re\left\{-\frac{zf'(z)}{f(z)}\right\}\geq\frac{1-(2-l)r+((2t-1)-(1+t)l)r^2+tlr^3}{(1-tr)(1-r^2)}.$$

*The bounds in* (2.1) *and* (2.2) *are sharp.*

*Proof.* Let $f\in\Gamma^r(\alpha,\beta)$; then there exists a function $g(z)=1/z+\sum_{n=0}^{\infty}b_nz^n\in\Sigma^*(r)$ such that $f/g$ satisfies the inequality (1.5) for some $\alpha,\beta(0\leq\alpha<1,0<\beta\leq1)$ and for all $z$ in $E$. Therefore by an application of Schwarz's lemma

$$(2.3)\qquad \frac{f(z)}{g(z)}=\frac{1+t\omega(z)}{1+s\omega(z)},$$

where $s=2\beta-1$, $t=2\alpha\beta-1$ and $\omega\in B$. Taking the logarithmic derivatives of (2.3) and then equating the real parts we get

$$(2.4)\ \mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}=\mathrm{Re}\left\{-\frac{zg'(z)}{g(z)}\right\}+(s-t)\mathrm{Re}\left\{\frac{z\omega'(z)}{(1+s\omega(z))(1+t\omega(z))}\right\}.$$

An application of Caratheodory's inequality [1].

$$|\phi'(z)|\leq\frac{1-|\phi(z)|^2}{1-|z|^2}$$

yields

$$(2.5)\qquad |z\omega'(z)-\omega(z)|\leq\frac{|z|^2-|\omega(z)|^2}{1-|z|^2},\quad \omega(z)=z\phi(z)\in B,\quad z\in E.$$

Applying (2.5) to the second term of the right hand side of (2.4), we find that

$$(2.6)\ \mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}\geq\mathrm{Re}\left\{-\frac{zg'(z)}{g(z)}\right\}-\frac{1}{s-t}\mathrm{Re}\left\{sp(z)+\frac{t}{p(z)}-(s+t)\right\}-\frac{r^2|sp(z)-t|^2-|1-p(z)|^2}{(s-t)(1-r^2)|p(z)|}$$

where $p(z)=(1+t\omega(z))/(1+s\omega(z))$. Since $g(z)=1/z+\sum_{n=0}^{\infty}a_nz^n\in\Sigma^*(r)$, it is known [2] that

$$(2.7)\qquad \mathrm{Re}\left\{-\frac{zg'(z)}{g(z)}\right\}\geq\frac{1+lr}{1+r},\quad l=2r-1.$$

Combining (2.6) and (2.7) we obtain

$$(2.8)\qquad \mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}\geq\frac{1+lr}{1+r}+\frac{s+t}{s-t}-\frac{1}{s-t}\left\{\mathrm{Re}\left(sp(z)+\frac{t}{p(z)}\right)+\frac{r^2|sp(z)-t|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\right\}.$$

Put $p(z)=\mathrm{R}e^{i\theta}$ and denote the right hand side of (2.8) by $J(R,\theta)$. Then

$$J(R,\theta)=\frac{1+lr}{1+r}+\frac{s+t}{s-t}-\frac{1}{s-t}\left\{\left(sR+\frac{t}{R}\right)\cos\theta + \frac{(1-s^2r^2)(d^2-a^2-R^2+2aR\cos\theta)}{(1-r^2)R}\right\}, \tag{2.9}$$

where

$$a=\frac{1-str^2}{1-s^2r^2},\quad d=\frac{(s-t)r}{1-s^2r^2}.$$

Now

$$\frac{\partial J(R,\theta)}{\partial\theta}=\frac{N(R)\sin\theta}{s-t}, \tag{2.10}$$

where

$$N(R)=sR+\frac{t}{R}+\frac{(1-s^2r^2)2a}{1-r^2} \geq -\left(R+\frac{1}{R}\right)+2a\left(\frac{1-s^2r^2}{1-r^2}\right)$$

for all $s, t$ $(-1\leq t<s\leq 1)$.
The minimum of

$$-\left(R+\frac{1}{R}\right)+2a\left(\frac{1-s^2r^2}{1-r^2}\right)$$

$a-d\leq R\leq a+d$, is attained at either $R=a+d$ or $R=a-d$ and it is easy to check that the values of this expsession at $R=a-d$ or $R=a+d$ are both non-negative. Thus $N(R)\geq 0$ for all $s, t$ satisfying $-1\leq t<s\leq 1$; and therefore it follows from (2.10) that the minimum of $J(R,\theta)$ is attained at $\theta=0$. Setting $\theta=0$ in (2.9) we get

$$J(R)\equiv J(R,O)=\frac{1+lr}{1+r}+\frac{s+t}{s-t}-\frac{1}{s-t}\left\{\left(s-\frac{1-s^2r^2}{1-r^2}\right)R + \left(t-\frac{1-t^2r^2}{1-r^2}\right)R^{-1}+\frac{2(1-str^2)}{1-r^2}\right\}. \tag{2.11}$$

which yields

$$\frac{dJ(R)}{dR}=-\frac{1}{s-t}\left\{\left(s-\frac{1-s^2r^2}{1-r^2}\right)-\frac{1}{R^2}\left(t-\frac{1-t^2r^2}{1-r^2}\right)\right\}.$$

We see that the absolute minimum of $J(R)$ occurs at the point

$$R_1=\left\{\frac{(1-t)(1+tr^2)}{(1-s)(1+sr^2)}\right\}^{1/2} \tag{2.12}$$

and the minimum value of $J(R)$ is given by

$$J(R_1)=\frac{1+lr}{1+r}+\frac{s+t}{s-t}-\frac{2(1-str^2)}{(s-t)(1-r^2)} + \frac{2\sqrt{(1-s)(1-t)(1+sr^2)(1+tr^2)}}{(s-t)(1-r^2)} \tag{2.13}$$

for $-1\leq t<s<1$, $-1\leq l<1$.

Next, we can check that $R_1>a-d$ but $R_1$ may not always be less than $a+d$. For the case when $R_1\notin[a-d, a+d]$, the absolute minimum of $J(R)$ on the segment $[a-d, a+d]$ occurs at

(2.14) $$R_2=a+d=\frac{1-tr}{1-sr}$$

and the minimum value is given by

(2.15) $$J(R_2)=\frac{1+lr}{1+r}-\frac{(s-t)r}{(1-sr)(1-tr)}$$

for $-1\leq t<s\leq 1$ and $-1\leq l<1$. The two minima given by (2.13) and (2.15) coincide for such values of $s, t$ $(-1\leq t<s<1)$ for which

(2.16) $$R_1=R_2.$$

The inequality (2.1) follows from (2.13) and (2.15). Furthermore (2.2) is obtained from (2.15) on fixing $s=1$. This completes the proof of the theorem.

For the sharpness of (2.2) and the second inequality in (2.1), we choose

(2.17) $$f(z)=\frac{1-tz}{1-sz}g(z)$$

with

$$g(z)=\frac{(1+z)^{(1-l)}}{z}$$

for $-1\leq t<s\leq 1$, $-1\leq l<1$ and $z\in E$. The equality sign in the first inequality in (2.1) occurs for the function

(2.18) $$f(z)=\frac{1-(1+t)bz+tz^2}{1-(1+s)bz+sz^2}g(z)$$

for $g$ given above, $-1\leq t<s<1$, $-1\leq l<1$, $z\in E$ and $b$ is determined by the equation

(2.19) $$\frac{1-(1+t)br+tr^2}{1-(1+s)br+sr^2}=R_1=\left\{\frac{(1-t)(1+tr^2)}{(1-s)(1+sr^2)}\right\}^{\frac{1}{2}}.$$

THEOREM 2. *Let* $f(z)=1/z+\sum_{n=0}^{\infty}a_nz^n$ *belongs to* $\Gamma^r(\alpha,\beta)$, *with* $g(z)=1/z+\sum_{n=0}^{\infty}b_zz^n\in\Sigma^*(r)$. *Let* $s, t, l, R_1$ *and* $R_2$ *are as given in the statement of* Theorem 1. *Let* $r_0=r_0(s,t)$ *be the unique positive root in* $(0,1]$ *of the equation*

(2.20) $$str^4-2str^3+(2(s+t)-1-st)r^2-2r+1=0.$$

*Then, we have the following*:

(i) *for* $0 \leq r \leq r_0$, *f is starlike in* $|z| < r_1$, *where* $r_1$ *is the smallest positive root in* $(0, 1)$ *of the equation*

$$(2.21) \quad \{(1+l^2)(s-t)+2l(s(1-t)+t(1-s))\}r^4-2(1-l)\{2st-s(1+l) -t(1-l)\}r^3+\{(1-l)^2(s-t)-4(1-s)(t-l)\}r^2 +4(1-s)(1-l)r-4(1-s)=0, \ R_1 \leq R_2, \quad (s \neq 1)$$

(ii) *for* $r_0 \leq r < 1$, *f is starlike in* $|z| < r_2$, *where* $r_2$ *is the smallest positive root in* $(0, 1)$ *of the equation*

$$(2.22) \quad stlr^3+\{s(1+t)+3t-l(s+t)\}r^2+(l-2s)r+1=0, \quad R_1 \geq R_2.$$

*Furthermore, if* $s=1$, *then f is starlike in* $0<|z|<r_3$, *where* $r_3$ *is the smallest positive root in* $(0, 1)$ *of the equation*

$$(2.23) \qquad tlr^3+((2t-1)-(1+t)l)r^2-(2-l)r+1=0.$$

*All these bounds are sharp with the extremal function given in* Theorem 1.

*Proof.* In view of (2.1), the radii of starlikeness for the functions in the class $\Gamma^r(\alpha, \beta)$, $\alpha \in [0, 1)$, $\beta \in (0, 1]$ can be determined from the following equations;

$$(2.24) \qquad \frac{1+lr}{1+r}+\frac{s+t}{s-t}-\frac{2(1-str^2)}{(s-t)(1-r^2)} +\frac{2\sqrt{(1-s)(1-t)(1+sr^2)(1+tr^2)}}{(s-t)(1-r^2)}=0$$

if $R_1 \leq R_2$, and

$$(2.25) \qquad (1+lr)(1-sr)(1-tr)-(s-t)r(1+r)=0$$

if $R_1 \geq R_2$. Equations (2.24) and (2.25) reduce, respectively, to the equations (2.21) and (2.22). The equation $R_1=R_2$ yields the equation (2.20). Further, $s=1$ in the equation (2.22) gives (2.23). This completes the proof of the theorem.

## 3. Some Applications

The case $s=1$, $t=2\alpha-1$ in the preceding theorem has been covered by Padmanabhan [2]. Putting $t=-1=l$ in (2.23) we get the equation $r^3-3r^2-3r+1=0$ which yields $r=2-\sqrt{3}=0.268\cdots$. Thus we get the following result.

COROLLARY 1. *If* $f(z)=1/z+\sum_{n=0}^{\infty} a_n z^n$ *belongs to* $\Gamma(0)$ *with* $g \in \Sigma^*(0)$, *then f is starlike in* $|z|<2-\sqrt{3}$. *The bound is sharp with the extremal function*

$$f(z)=\frac{(1+z)^3}{z(1-z)}.$$

*Replacing s by o and t by $-\beta$ in* Theorem 2 *we easily obtain the following result.*

COROLLARY 2. *Let $f(z)=1/z+\sum_{n=0}^{\infty} a_n z^n$ be analytic in $E_0$. If f satisfies the condition*

$$\left|\frac{f(z)}{g(z)}-1\right|<\beta, \quad (0<\beta\leq 1)$$

*for some $g\in\Sigma^*(r)$ and for all $z\in E$, then we have the following:*
(i) *If $\beta\geq\beta_0$, where $\beta_0$ is the smallest positive root of*

$$2\gamma^2\beta^2-2\beta(2\gamma^2+\gamma+1)+3-4\gamma^2=0,$$

*then f is univalent and starlike in*

$$0<|z|<\frac{2}{\sqrt{\{(2\gamma-1)^2+8\beta(2-\gamma)\}}-(2\gamma-1)},$$

(ii) *and if $\beta\leq\beta_0$, f is starlike in $0<|z|<r_2$, where $r_2$ is the smallest positive root of the equation*

$$\beta(1-r)^2r^4-2(1-r)^2\beta r^3+\beta(r^2+6r-2)r^2+2(1-r)r-1=0.$$

*All these bounds in $|z|$ are sharp.*

For $s=1-2\alpha$ and $l=-1=t$, Theorem 2 gives the following result.

COROLLARY 3. *Let $f(z)=1/z+\sum_{n=0}^{\infty} a_n z^n$ be analytic in $E_0$. Let $f\in\Lambda^0(\alpha)$, and let $r_0$ be the unique positive root in $(0,1]$ of the equation*

$$(2\alpha-1)r^3-3(2\alpha-1)r^2-3r+1=0.$$

*Then*
(i) *for $0\leq r\leq r_0$, f is starlike in $|z|<r_1$, where $r_1$ is the smallest positive root of the equation*

$$2(1-\alpha)r^4-4(3\alpha-1)r^3+2(3\alpha+1)r^2+8\alpha r-8\alpha=0,$$

(ii) *and for $r_0\leq r<1$, f is starlike in $0<|z|<r_2$, where*

$$r_2=\frac{1}{\sqrt{(4\alpha^2+3)}+(2\alpha-1)}.$$

*These bounds are sharp.*

Our next result is obtained from Theorem 2 on the replacement of $s$ by $(\delta-1)/\delta$, $(\delta>1/2)$, $t$ and $l$ by $-1$.

COROLLARY 4. *Let $f(z)=1/z+\sum_{n=0}^{\infty} a_n z^n$ be analytic in $E_0$. Let $f\in G^0(\delta)$ and let $r_0$ be the unique positive root in $(0,1]$ of the equation*

$$-qr^4+2qr^3-3(1-q)r^2-2r+1=0.$$

*Then*

(i) *for* $0 \leq r \leq r_0$, *f is starlike in* $|z| < r_1$, *where* $r_1$ *is the smallest positive root of the equation*

$$(1+q)r^4+4qr^3+(5-3q)r^2+4(1-q)r-4(1-q)=0,$$

(ii) *and for* $r_0 \leq r < 1$, *f is starlike in*

$$0<|z|<\frac{1}{q+\sqrt{((q+1)^2-1)}}$$

*where* $q=1-1/\delta$. *These bounds are sharp.*

Finally, we remark that the radii of starlikeness for the functions in the class $\Gamma^r_{\alpha,\beta}$ can be obtained from Theorem 2 by replacing $s$ by $\beta, t$ by $\beta(2\alpha-1)$.

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