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THE RELATIONSHIP BETWEEN $\mathrm{DIM}_A(E)$ AND $\mathrm{DIM}_B(E)$

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Introduction

Let B be a finite integral extension of a commutative ring A with identity and E a finite B-module. The purpose of this note is to study the relationship between $\dim_A(E)$ and $\dim_B(E)$ where dim means the Krull's one. The ring and dim used here will be commuative with identity and Krull's one respectively.

1. Preliminaries

Let B be integral over A. If $\mathfrak{b}$ is an ideal of B and $\mathfrak{A}=A\cap\mathfrak{b}$, then $B/\mathfrak{b}$ is integral over $A/\mathfrak{A}$.

In fact for any $x\in B$, we have, say $x^n+a_1x^{n-1}+\cdots+a_n=0$ with $a_i\in A$ if and only if $\bar{x}^n+\bar{a}_1\bar{x}^{n-1}+\cdots+\bar{a}_n=0$ (mod $\mathfrak{b}$) so that $B/\mathfrak{b}$ is integral over $A/\mathfrak{A}$.

Moreover let A, B be as before. Then the pair A, B satisfies incomparability and going-up [2, p. 29].

If the rings A, B satisfy going-up and incomparability then $\dim(B)$ equals $\dim(A)$ [2, p. 31] and $\dim(A/\mathfrak{A})$ therefore equals $\dim(B/\mathfrak{b})$.

Let spec(A) be the set of all prime ideals of a ring A. For each subset S of A, let $V(S)$ denote the set of all prime ideals of A which contains S. Let E be an A-module.

The support of E is defined to be the set supp(E) of prime ideals $\mathfrak{p}$ of A such that $E_\mathfrak{p}\neq 0$ and ann(E) to be the set of all $a\in A$ such that $aE=0$.

PROPOSITION. *Let A be a ring and E an A-module. The following results hold*:

i) $E=\sum E_i \Longrightarrow \mathrm{supp}(E)=\bigcup \mathrm{supp}(E_i)$

ii) *If E is finitely generated, then* $\mathrm{supp}(E)=V(\mathrm{ann}E)$.

(*and therefore a closed subset of* spec(A)).

Proof. i) $\mathfrak{p}\in\mathrm{supp}(E)\Longrightarrow E_\mathfrak{p}=(\sum E_i)_\mathfrak{p}\neq 0$ implies that $(E_i)_\mathfrak{p}\neq 0$ for at

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least one i. Hence $\mathfrak{p}\in \bigcup \operatorname{supp}(E_i)$. Reverse inclusion is obvious.

ii) Let $\{x_1, x_2, \ldots\ldots, x_n\}$ be generators of E and $E_i=Ax_i$, Then $A/\mathscr{T}_i=E_i$ where $\mathscr{T}_i=\operatorname{ann}(x_i)$. Therefore $\operatorname{supp}(E_i)=V(\mathscr{T}_i)$. By i) $\operatorname{supp}(E)=\bigcup_{i=1}^{n}$ (supp $E_i)=\bigcup_{i=1}^{n} V(\mathscr{T}_i)=V(\bigcup \mathscr{T}_i)=V(\operatorname{ann}E)$.

2. Main theorem

THEOREM. *Let A, B, E be as in the introduction. Then* $\dim_A(E)$ *equals* $\dim_B(E)$.

Proof. $\dim_B(E)=\sup\{\dim B/\mathfrak{p} \mid \mathfrak{p}\in\operatorname{spec}(B), E_\mathfrak{p}\neq 0\}$.

Let $n=\dim_B(E)$ and $\mathfrak{p}\in\operatorname{spec}(B)$ be such that $E_\mathfrak{p}\neq 0$ and $\dim B/\mathfrak{p}=n$. Put $P=\mathfrak{p}\cap A$, then $B/\mathfrak{p}$ is an integral extension of A/P, hence $\dim B/\mathfrak{p}=\dim A/P$.

Moreover $E_\mathfrak{p}$ is a localization of $EP=(A-P)^{-1}E$, therefore $E_P\neq 0$, so $\dim_A E\geqq n=\dim_B E$.

To prove the converse let $P\in\operatorname{spec}(A)$ be such that $\dim (A/P)=\dim_A(E)$ and $E_P\neq 0$. We have to prove that there exists $\mathfrak{p}\in\operatorname{spec}(B)$ lying over P such that $E_P\neq 0$. Replacing A, B, E by A_P, B_P, E_P, we may suppose that (A, P) is a lacal ring and $E\neq 0$, Then the prime ideals of B lying over P are exactly the maximal ideals of B, and since $\operatorname{supp}_B(E)$ is a closed subset by proposition there exists a maximal ideal $\mathfrak{p}$ such that $E_\mathfrak{p}\neq 0$.

COROLLARY. *Let A, B, E be as in theorem, $\mathscr{C}_B$ the category of finite B-modules and* dim: $\mathscr{C}_B\rightarrow N$ *to the* Krull *dimension. Then the followings are satisfied*:

i) $\dim_A(B/\mathfrak{M})=0$ *where $\mathfrak{M}$ is a maximal ideal of B.*

ii) *if* $0\longrightarrow E'\longrightarrow E\longrightarrow E''\longrightarrow 0$ *is an exact sequence of $\mathscr{C}_B$ then* $\dim_B(E)=\max(\dim_A(E'), \dim_A(E''))$.

iii) *if* $(A, \mathfrak{M})$ *is a local ring and* $0\longrightarrow E\longrightarrow E\longrightarrow E/mE\longrightarrow 0$ *is an exact sequence of $\mathscr{C}_B$ where $m\in\mathfrak{M}$ then* $\dim_A(E)=1+\dim_A(E/mE)$.

Proof. i), ii) are clear by theorem and for Proof of iii) see [3].

References

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