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ON CERTAIN SUBCLASSES OF ANALYTIC P-VALENT FUNCTIONS

SHIGEYOSHI OWA*

1. Introduction

Let $\mathcal{S}_p$ denote the class of functions

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n}z^{p+n} \qquad (p\in\mathcal{N})$$

analytic and p-valent in the unit disk $\mathcal{U}=\{|z|<1\}$. We say that $f(z)$ belongs to the class $\mathcal{S}_p(a,b)$ if $f(z)\in\mathcal{S}_p$ satisfies the condition

$$\left|\frac{\dfrac{f'(z)}{pz^{p-1}}-1}{\dfrac{f'(z)}{pz^{p-1}}+(1-2a)}\right|<b \qquad (z\in\mathcal{U})$$

for $a(0\leqq a<1)$ and $b(0<b\leqq 1)$. Further $f(z)$ is said to belong to the class $\mathcal{K}_p(a,b)$ if $zf'(z)/p\in\mathcal{S}_p(a,b)$. Recently S. Owa [10] showed a distortion theorem, coefficient estimates and a radius of convexity for the class $\mathcal{S}_p(a, b)$.

In particular, the class $\mathcal{S}_1(0,b)$ was studied by K. S. Padmanabhan [14] and later by T. R. Caplinger and W. M. Causey [4]. Furthermore S. Owa [11], [12] studied the class $S_1(a,b)$.

Let $\mathcal{T}_p$ denote the subclass of $\mathcal{S}_p$ consisting of functions analytic and p-valent which can expressed in the form

$$f(z)=z^p-\sum_{n=1}^{\infty} a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ p\in\mathcal{N}).$$

We denote by $\mathcal{T}_p(a, b)$ and $\mathcal{C}_p(a, b)$ the classes obtained by taking intersections of the classes $\mathcal{S}_p(a,b)$ and $\mathcal{K}_p(a,b)$ with $\mathcal{T}_p$, respectively.

In 1976, V. P. Gupta and P. K. Jain [7] studied the class $\mathcal{T}_1(a, b)$. Moreover H. Silverman [17], H. Silverman and E. M. Silvia [18], [19] and O. P. Ahuja and P. K. Jain [2] have studied certain subclasses of univalent functions with negative coefficients. For other classes of analytic p-valent functions with negative coefficients, R. M. Goel and N. S. Sohi

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[6], R. M. Goel and S. Owa [5] and H. M. Srivastava and S. Owa [20] showed some results.

2. Coefficient estimates

THEOREM 1. *A function*

$$f(z)=z^p-\sum_{n=1}^{\infty} a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ p\in\mathcal{N})$$

is in the class $\mathcal{C}_p(a, b)$ *if, and only if,*

$$\sum_{n=1}^{\infty}(p+n)(1+b)a_{p+n}\leqq 2b(1-a)p.$$

This result is sharp.

Proof. Let $|z|=1$. Then we get

$$\begin{aligned}
&|f'(z)-pz^{p-1}|-b|f'(z)+(1-2a)pz^{p-1}|\\
&\quad=\left|\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n-1}\right|\\
&\qquad-b\left|2(1-a)pz^{p-1}-\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n-1}\right|\\
&\quad\leqq\sum_{n=1}^{\infty}(p+n)(1+b)a_{p+n}-2b(1-a)p\\
&\quad\leqq 0.
\end{aligned}$$

Hence, by the maximum modulus theorem, $f(z)$ is in the class $\mathcal{C}_p(a, b)$.

On the other hand, assume that

$$\left|\frac{\dfrac{f'(z)}{pz^{p-1}}-1}{\dfrac{f'(z)}{pz^{p-1}}+(1-2a)}\right|<b \qquad (z\in\mathcal{U}).$$

Since $|\mathrm{Re}(z)|\leqq|z|$ for any z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n-1}}{2(1-a)pz^{p-1}-\sum_{n=1}^{\infty}(p+n)a_{p+n}z^{p+n-1}}\right\}\leqq b. \tag{1}$$

Choose values of z on the real axis so that $f'(z)/z^{p-1}$ is real. Upon clearing the denominator in (1) and letting $z\to 1$ through real values, we get

$$\sum_{n=1}^{\infty}(p+n)a_{p+n}\leqq 2b(1-a)p-\sum_{n=1}^{\infty}b(p+n)a_{p+n}$$

which implies that

$$\sum_{n=1}^{\infty}(p+n)(1+b)a_{p+n}\leqq 2b(1-a)p.$$

The function

$$f(z)=z^p-\frac{2b(1-a)p}{(p+n)(1+b)}z^{p+n}$$

is an extremal function.

COROLLARY 1. *Let a function*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in\mathcal{N})$$

be in the class $\mathcal{T}_p(a,b)$. *Then we have*

$$a_{p+n}\leqq\frac{2b(1-a)p}{(p+n)(1+b)}$$

for any $n\geqq 1$. *The equality holds for the function*

$$f(z)=z^p-\frac{2b(1-a)p}{(p+n)(1+b)}z^{p+n}.$$

THEOREM 2. *A function*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ p\in\mathcal{N})$$

is in the class $\mathcal{C}_p(a,b)$ *if, and only if,*

$$\sum_{n=1}^{\infty}(p+n)^2(1+b)a_{p+n}\leqq 2b(1-a)p^2.$$

This result is sharp.

Proof. The function $f(z)$ is in the class $\mathcal{C}_p(a,b)$ if, and only if, $zf'(z)/p\in\mathcal{T}_p(a,b)$. Now, since

$$\frac{zf'(z)}{p}=z^p-\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)a_{p+n}z^{p+n},$$

be replacing a_{p+n} by $\{(p+n)/p\}a_{p+n}$ in Theorem 1, we have the theorem.

COROLLARY 2. *Let a function*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ p\in\mathcal{N})$$

be in the class $\mathcal{C}_p(a,b)$. *Then we have*

$$a_{p+n}\leqq\frac{2b(1-a)p^2}{(p+n)^2(1+b)}$$

for any $n\geqq 1$. *The equality holds for the function*

$$f(z)=z^p-\frac{2b(1-a)p^2}{(p+n)^2(1+b)}z^{p+n}$$

3. Some properties of $\mathcal{T}_p(a,b)$ and $\mathcal{C}_p(a,b)$

THEOREM 3. *Let* $0\leqq a'\leqq a''<1$ *and* $0<b'\leqq b''\leqq 1$. *Then we have*

$$\mathcal{T}_p(a',\ b'')\supset\mathcal{T}_p(a'',\ b').$$

Theorem 3 is clear from the definition of $\mathcal{T}_p(a,b)$.

THOREM 4. *Let* $0 \leqq a' \leqq a'' < 1$ *and* $0 < b' \leqq b'' \leqq 1$. *Then we have*

$$\mathcal{C}_p(a', b'') \supset \mathcal{C}_p(a'', b').$$

Proof. Let a function

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \geqq 0, \quad p \in \mathcal{N})$$

be in the class $\mathcal{C}_p(a'', b')$ and $b'' = b' + \varepsilon$. Since

$$\sum_{n=1}^{\infty} (p+n)^2 (1+b') a_{p+n} \leqq 2b'(1-a'')p^2$$

by Theorem 2, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} (p+n)^2 (1+b'') a_{p+n} \\ &= \sum_{n=1}^{\infty} (p+n)^2 (1+b'+\varepsilon) a_{p+n} \\ &= \sum_{n=1}^{\infty} (p+n)^2 (1+b') a_{p+n} + \varepsilon \sum_{n=1}^{\infty} (p+n)^2 a_{p+n} \\ &\leqq 2b'(1-a'')p^2 + \frac{2\varepsilon b'(1-a'')p^2}{1+b'} \\ &\leqq 2b'(1-a'')p^2 + 2\varepsilon b'(1-a'')p^2 \\ &= 2(b'+\varepsilon)(1-a'')p^2 \\ &\leqq 2b''(1-a')p^2 \end{aligned}$$

which gives that $f(z) \in \mathcal{C}_p(a, b'')$.

THEOREM 5. *Let a function*

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \geqq 0, \quad p \in \mathcal{N})$$

be in the class $\mathcal{C}_p(a, b)$. *Then* $f(z)$ *belongs to the class* $\mathcal{T}_p((ap+1)/(p+1), b)$, *that is*

$$\mathcal{C}_p(a, b) \subset \mathcal{T}_p\left(\frac{ap+1}{p+1}, b\right).$$

Proof. Since $f(z) \in \mathcal{C}_p(a, b)$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} (p+n)(1+b) a_{p+n} &\leqq \frac{2b(1-a)p^2}{p+1} \\ &= 2b\left(1 - \frac{ap+1}{p+1}\right)p \end{aligned}$$

with the aid of Theorem 2. Further

$$0 < \frac{ap+1}{p+1} < 1$$

for $0 \leqq a < 1$ and $p \in \mathcal{N}$. Consequently we have the theorem with Theorem 1.

4. Distortion theorems

THEOREM 6. *Let a function*

$$f(z)=z^p-\sum_{n=1}^{\infty} a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ p\in\mathcal{N})$$

be in the class $\mathcal{T}_p(a,b)$. *Then we have*

$$|f(z)|\geqq|z|^p-\frac{2b(1-a)p}{(p+1)(1+b)}|z|^{p+1}$$

and

$$|f(z)|\leqq|z|^p+\frac{2b(1-a)p}{(p+1)(1+b)}|z|^{p+1}$$

for $z\in\mathcal{U}$. *Further*

$$|f'(z)|\geqq p|z|^{p-1}-\frac{2b(1-a)p}{1+b}|z|^p$$

and

$$|f'(z)|\leqq p|z|^{p-1}+\frac{2b(1-a)p}{1+b}|z|^p$$

for $z\in\mathcal{U}$. *These estimates are sharp.*

Proof. By using Theorem 1, we obtain

$$(p+1)(1+b)\sum_{n=1}^{\infty}a_{p+n}\leqq\sum_{n=1}^{\infty}(p+n)(1+b)a_{p+n}\leqq 2b(1-a)p$$

which implies that

$$\sum_{n=1}^{\infty}a_{p+n}\leqq\frac{2b(1-a)p}{(p+1)(1+b)}\,.$$

Consequently we have

$$\begin{aligned}|f(z)|&\geqq|z|^p-\sum_{n=1}^{\infty}a_{p+n}|z|^{p+n}\\&\geqq|z|^p-|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}\\&\geqq|z|^p-\frac{2b(1-a)p}{(p+1)(1+b)}|z|^{p+1}\end{aligned}$$

and

$$\begin{aligned}|f(z)|&\leqq|z|^p+\sum_{n=1}^{\infty}a_{p+n}|z|^{p+n}\\&\leqq|z|^p+|z|^{p+1}\sum_{n=1}^{\infty}a_{p+n}\\&\leqq|z|^p+\frac{2b(1-a)p}{(p+1)(1+b)}|z|^{p+1}\end{aligned}$$

for $z\in\mathcal{U}$.

In order to show the second half of the theorem, by using

$$\sum_{n=1}^{\infty}(p+n)a_{p+n}\leqq\frac{2b(1-a)p}{1+b},$$

we obtain

$$\begin{aligned}|f'(z)| &\geqq p|z|^{p-1}-\sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^{p+n-1}\\ &\geqq p|z|^{p-1}-|z|^{p}\sum_{n=1}^{\infty}(p+n)a_{p+n}\\ &\geqq p|z|^{p-1}-\frac{2b(1-a)p}{1+b}|z|^{p}\end{aligned}$$

and

$$\begin{aligned}|f'(z)| &\leqq p|z|^{p-1}+\sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^{p+n-1}\\ &\leqq p|z|^{p-1}+|z|^{p}\sum_{n=1}^{\infty}(p+n)a_{p+n}\\ &\leqq p|z|^{p-1}+\frac{2b(1-a)p}{1+b}|z|^{p}\end{aligned}$$

for $z\in\mathcal{U}$. The bounds are sharp and are attained for the function

$$f(z)=z^{p}-\frac{2b(1-a)p}{(p+1)(1+b)}z^{p+1}.$$

COROLLARY 3. *Under the hypotheses of Theorem* 6, *$f(z)$ is included in the disk with center at the origin and radius* $1+2b\ (1-a)\ p/(p+1)\ (1+b)$. *Further $f'(z)$ is included in the disk with center at the origin and radius* $p(1+3b-2ab)/(1+b)$.

THEOREM 7. *Let a function*

$$f(z)=z^{p}-\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\qquad (a_{p+n}\geqq 0,\ p\in\mathcal{N})$$

be in the class $\mathcal{C}_p(a,\ b)$. Then we have

$$|f(z)|\geqq|z|^{p}-\frac{2b(1-a)p^{2}}{(p+1)^{2}(1+b)}|z|^{p+1}$$

and

$$|f(z)|\leqq|z|^{p}+\frac{2b(1-a)p^{2}}{(p+1)^{2}(1+b)}|z|^{p+1}$$

for $z\in\mathcal{U}$. Further

$$|f'(z)|\geqq p|z|^{p-1}-\frac{2b(1-a)p^{2}}{(p+1)\ (1+b)}|z|^{p}$$

and

$$|f'(z)|\leqq p|z|^{p-1}+\frac{2b(1-a)p^{2}}{(p+1)\ (1+b)}|z|^{p}$$

for $z\in\mathcal{U}$. If $p\in\mathcal{N}-\{1\}$, then we have

$$|f''(z)|\geqq p(p-1)|z|^{p-2}-\frac{2b(1-a)p^{2}}{1+b}|z|^{p-1}$$

and

$$|f''(z)|\leqq p(p-1)|z|^{p-2}+\frac{2b(1-a)p^2}{1+b}|z|^{p-1}$$

for $z\in\mathcal{U}$. *The estimates for* $f(z)$ *and* $f'(z)$ *are sharp and are attained for the function*

$$f(z)=z^p-\frac{2b(1-a)p^2}{(p+1)^2(1+b)}z^{p+1}.$$

Proof. The proofs for $|f(z)|$ and $|f'(z)|$ are obtained by using the same technique as in the proof of Theorem 6 with the aid of Theorem 2. Further, for $p\in\mathcal{N}-\{1\}$ and $z\in\mathcal{U}$, we have

$$\begin{aligned}|f''(z)|&\geqq p(p-1)|z|^{p-2}-\sum_{n=1}^{\infty}(p+n)(p+n-1)a_{p+n}|z|^{p+n-2}\\&\geqq p(p-1)|z|^{p-2}-|z|^{p-1}\sum_{n=1}^{\infty}(p+n)^2a_{p+n}\\&\geqq p(p-1)|z|^{p-2}-\frac{2b(1-a)p^p}{1+b}|z|^{p-1}\end{aligned}$$

and

$$\begin{aligned}|f''(z)|&\leqq p(p-1)|z|^{p-2}+\sum_{n=1}^{\infty}(p+n)(p+n-1)a_{p+n}|z|^{p+n-2}\\&\leqq p(p-1)|z|^{p-2}+|z|^{p-1}\sum_{n=1}^{\infty}(p+n)^2a_{p+n}\\&\leqq p(p-1)|z|^{p-2}+\frac{2b(1-a)p^2}{1+b}|z|^{p-1}\end{aligned}$$

by using Theorem 2.

COROLLARY 4. *Under the hypotheses of Theorem 7,* $f(z)$ *is included in the disk with center at the origin and radius* $1+2b(1-a)p^2/(p+1)^2(1+b)$, *and* $f'(z)$ *is included in the disk with center at the origin and radius* $p+2b(1-a)p^2/(p+1)(1+b)$. *Further* $f''(z)$ *is included in the disk with center at the origin and radius* $p(p-1)+2b(1-a)p^2/(1+b)$.

4. Closure theorems

THEOREM 8. *Let*

$$f_i(z)=z^p-\sum_{n=1}^{\infty}a_{i,p+n}z^{p+n}\qquad(a_{i,p+n}\geqq 0,\ p\in\mathcal{N})$$

be in the classes $\mathcal{T}_p(a_i,b_i)$ *for each* $i=1,2,3,\dots,m$. *Then the function*

$$h(z)=z^p-\frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m}a_{i,p+n}\right)z^{p+n}$$

is in the class $\mathcal{T}_p(a,b)$, *where* $a=\operatorname*{Min}_{1\leq i\leq m}\{a_i\}$ *and* $b=\operatorname*{Max}_{1\leq i\leq m}\{b_i\}$.

Proof. Since $f_i(z)\in\mathscr{T}_p(a_i, b_i)$ for each $i=1, 2, 3, \dots, m,$ we have

$$\sum_{n=1}^{\infty}(p+n)(1+b_i)a_{i,p+n}\leqq 2b_i(1-a_i)p$$

by Theorem 1. Hence we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}&\left(\frac{p+n}{p}\right)\left(\frac{1}{m}\sum_{i=1}^{m}a_{i,p+n}\right)\\ &=\frac{1}{m}\sum_{n=1}^{\infty}\left\{\sum_{i=1}^{m}\left(\frac{p+n}{p}\right)a_{i,p+n}\right\}\\ &\leqq\frac{1}{m}\sum_{i=1}^{m}\left\{\frac{2b_i(1-a_i)}{1+b_i}\right\}\\ &\leqq\frac{2b(1-a)}{1+b},\end{aligned}$$

because $2b'(1-a')/(1+b')\geqq 2b''(1-a'')/(1+b'')$ for $a'\leqq a''$ and $b'\geqq b''$. Thus we get

$$\sum_{n=1}^{\infty}(p+n)\ (1+b)\left(\frac{1}{m}\sum_{i=1}^{m}a_{i,p+n}\right)\leqq 2b(1-a)p$$

which shows that $h(z)\in\mathscr{T}_p(a, b)$.

THEOREM 9. *Let*

$$f_i(z)=z^p-\sum_{n=1}^{\infty}a_{i,p+n}z^{p+n}\qquad (a_{i,p+n}\geqq 0,\ p\in\mathscr{N})$$

be in the classes $\mathscr{C}_p\ (a_i, b_i)$ *for each* $i=1, 2, 3, \dots, m$. *Then the function*

$$h(z)=z^p-\frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m}a_{i,p+n}\right)z^{p+n}$$

is in the class $\mathscr{C}_p(a, b)$, *where* $a=\operatorname*{Min}_{1\leqq i\leqq m}\{a_i\}$ *and* $b=\operatorname*{Max}_{1\leqq i\leqq m}\{b_i\}$.

The proof of Theorem 9 is obtained by using the same technique as in the proof of Theorem 8 with the aid of Theorem 2.

THEOREM 10. *Let*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\qquad (a_{p+n}\geqq 0,\ p\in\mathscr{N})$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n}\qquad (b_{p+n}\geqq 0,\ p\in\mathscr{N})$$

be in the classes $\mathscr{T}_p(a, b)$ *and* $\mathscr{C}_p(a, b)$, *respectively. Then the function*

$$\mathscr{H}(z)=z^p-\left(\frac{p+1}{2p+1}\right)\sum_{n=1}^{\infty}(a_{p+n}+b_{p+n})z^{p+n}$$

is in the class $\mathscr{T}_p(a, b)$.

Proof. Since $f(z)\in\mathscr{T}_p(a, b)$ and $g(z)\in\mathscr{C}_p(a, b)$, by using Theorem 1 and Theorem 2, we get

$$\sum_{n=1}^{\infty}(p+n)(1+b)a_{p+n}\leqq 2b(1-a)p$$

and

$$\sum_{n=1}^{\infty}(p+n)(1+b)\,b_{p+n}\leqq\frac{2b(1-a)p^2}{p+1}$$

Therefore we have

$$\left(\frac{p+1}{2p+1}\right)\sum_{n=1}^{\infty}(p+n)(1+b)(a_{p+n}+b_{p+n})\leqq 2b(1-a)p$$

which implies that $\mathscr{H}(z)\in\mathscr{T}_p(a,b)$.

THEOREM 11. *Let*

$$f_p(z)=z^p \qquad (p\in\mathscr{N})$$

and

$$f_{p+n}(z)=z^p-\frac{2b(1-a)p}{(p+n)(1+b)}z^{p+n} \qquad (p\in\mathscr{N})$$

for $n=1,2,3,\cdots$. *Then* $f(z)$ *belongs to the class* $\mathscr{T}_p(a,b)$ *if, and only if, it can be expressed in the form*

$$f(z)=\sum_{n=0}^{\infty}\lambda_{p+n}f_{p+n}(z),$$

where $\lambda_{p+n}\geqq 0$ *and*

$$\sum_{n=0}^{\infty}\lambda_{p+n}=1.$$

Proof. Assume that

$$f(z)=\sum_{n=0}^{\infty}\lambda_{p+n}f_{p+n}(z)$$

$$=z^p-\sum_{n=1}^{\infty}\frac{2b(1-a)p}{(p+n)(1+b)}\lambda_{p+n}z^{p+n}.$$

Then we obtain

$$\sum_{n=1}^{\infty}\left\{\lambda_{p+n}\frac{(p+n)(1+b)}{2b(1-a)p}\,\frac{2b(1-a)p}{(p+n)(1+b)}\right\}=\sum_{n=1}^{\infty}\lambda_{p+n}$$

$$=1-\lambda_p\leqq 1.$$

This shows that $f(z)\in\mathscr{T}_p(a,b)$ by Theorem 1.

On the other hand, let

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ p\in\mathscr{N})$$

be in the class $\mathscr{T}_p(a,b)$. Then, by using Corollary 1, we get

$$a_{p+n}\leqq\frac{2b(1-a)p}{(p+n)(1+b)}$$

for any $n\geqq 1$. On putting

$$\lambda_{p+n}=\frac{(p+n)(1+b)}{2b(1-a)p}a_{p+n} \qquad (n=1,2,3,\cdots)$$

and

$$\lambda_p=1-\sum_{n=1}^{\infty}\lambda_{p+n},$$

we have the expression

$$f(z)=\sum_{n=0}^{\infty}\lambda_{p+n}f_{p+n}(z).$$

This completes the proof of the theorem.

THEOREM 12. *Let*

$$f_p(z)=z^p \qquad (p\in\mathcal{N})$$

and

$$f_{p+n}(z)=z^p-\frac{2b(1-a)p^2}{(p+n)^2(1+b)}z^{p+n} \qquad (p\in\mathcal{N}).$$

for $n=1, 2, 3, \cdots$. *Then* $f(z)$ *belongs to the class* $\mathcal{C}_p(a, b)$ *if, and only if, it can be expressed in the form*

$$f(z)=\sum_{n=0}^{\infty}\lambda_{p+n}f_{p+n}(z),$$

where $\lambda_{p+n}\geqq 0$ *and*

$$\sum_{n=0}^{\infty}\lambda_{p+n}=1.$$

The proof of Theorem 12 is given in much the same way as Theorem 11.

5. Hadamard products

Let $f*g(z)$ denote the Hadamard product of two functions

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ \ p\in\mathcal{N})$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n} \qquad (b_{p+n}\geqq 0,\ \ p\in\mathcal{N}),$$

that is,

$$f*g(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}b_{p+n}z^{p+n}.$$

THEOREM 13. *Let the functions*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ \ p\in\mathcal{N})$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n} \qquad (b_{p+n}\geqq 0,\ \ p\in\mathcal{N})$$

be in the classes $\mathcal{T}_p(a', b')$ *and* $\mathcal{T}_p(a'', b'')$, *respectively. Then the Hadamard product* $f*g(z)$ *belongs to the class* $\mathcal{T}_p(a(2-a), b)$, *where* $a=\mathrm{Min}\{a', a''\}$ *and*

$b=\mathrm{Max}\{b', b''\}$.

Proof. Since $f(z)\in\mathcal{T}_p(a', b')$ and $g(z)\in\mathcal{T}_p(a'', b'')$, by using Theorem 1, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}(p+n)(1+b)a_{p+n}b_{p+n}\\&\qquad\leqq 2b(1-a)p\frac{2b_0(1-a)p}{(p+1)(1+b_0)}\\&\qquad\leqq 2b\{1-a(2-a)\}p,\end{aligned}$$

where $b_0=\mathrm{Min}\{b', b''\}$. Further $0\leqq a(2-a)<1$, because $0\leqq a<1$. Consequently the Hadamard product $f*g(z)$ is in the class $\mathcal{T}_p(a(2-a), b)$ by Theorem 1.

COROLLARY 5. *Under the hypotheses of* Theorem 13, *the Hadamard product* $f*g(z)$ *belongs to the class* $\mathcal{T}_p(a, b)$.

Proof, By using Theorem 3, we obtain

$$\mathcal{T}_p(a, b)\supset\mathcal{T}_p(a(2-a), b)$$

which implies that $f*g(z)\in\mathcal{T}_p(a, b)$ with Theorem 13.

THEOREM 14. *Let the functions*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\qquad(a_{p+n}\geqq 0,\ \ p\in\mathcal{N})$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n}\qquad(b_{p+n}\geqq 0,\ \ p\in\mathcal{N})$$

be in the classes $\mathcal{C}_p(a', b')$ *and* $\mathcal{C}_p(a'', b'')$, *respectively. Then the Hadamard product* $f*g(z)$ *belongs to the class* $\mathcal{C}_p(a(2-a), b)$, *where* $a=\mathrm{Min}\{a', a''\}$ *and* $b=\mathrm{Max}\{b', b''\}$.

The proof of Theorem 14 is obtained by using the same technique as in the proof of Theorem 13.

COROLLARY 6. *Under the hypotheses of* Theorem 14, *the Hadamard product* $f*g(z)$ *belongs to the class* $\mathcal{C}_p(a, b)$.

THEOREM 15. *Let the functions*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\qquad(a_{p+n}\geqq 0,\ \ p\in\mathcal{N})$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n}\qquad(b_{p+n}\geqq 0,\ \ p\in\mathcal{N})$$

be in the same class $\mathcal{T}_p(a, b)$ *with* $0\leqq a<1$ *and* $0<b\leqq\sqrt{2}/2$. *Then the Hadamard product* $f*g(z)$ *belongs to the class* $\mathcal{C}_p$ $(a(2-a), 2b^2)$.

Proof. Since $f(z)\in\mathscr{T}_p(a, b)$ and $g(z)\in\mathscr{T}_p(a, b)$, by using Theorem 1, we obtain

$$\sum_{n=1}^{\infty}(p+n)(1+b)a_{p+n}\leqq 2b(1-a)p$$

and

$$(p+n)(1+b)b_{p+n}\leqq 2b(1-a)p$$

for any $n\geqq 1$. Hence we have

$$\begin{aligned}&\sum_{n=1}^{\infty}(p+n)^2(1+2b^2)a_{p+n}b_{p+n}\\ &\leqq\sum_{n=1}^{\infty}(p+n)^2(1+b)^2a_{p+n}b_{n+n}\\ &\leqq 4b^2\{1-a(2-a)\}p^2.\end{aligned}$$

Further $0\leqq a(2-a)<1$ for $0\leqq a<1$ and $0<2b^2\leqq 1$ for $0<b\leqq\sqrt{2}/2$. This completes the proof of the theorem with Theorem 2.

THEOREM 16. *Let the functions*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\qquad (a_{p+n}\geqq 0,\ p\in\mathscr{N})$$

and

$$g(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n}\qquad (b_{p+n}\geqq 0,\ p\in\mathscr{N})$$

be in the same class $\mathscr{T}_p(a, b)$. *Then the Hadamard product* $f*g(z)$ *belongs to the class* $\mathscr{C}_p(a(2-a), b)$.

Proof. Since $f(z)\in\mathscr{T}_p(a, b)$ and $g(z)\in\mathscr{T}_p(a, b)$, by Theorem 1, we get

$$\begin{aligned}\sum_{n=1}^{\infty}(p+n)^2(1+b)a_{p+n}b_{p+n}&\leqq\frac{4b^2(1-a)^2p^2}{1+b}\\ &\leqq 2b\{1-a(2-a)\}p^2\end{aligned}$$

which gives that $f*g(z)\in\mathscr{C}(a(2-a), b)$.

6. Fractional calculus

There are many definitions of the fractional calculus, that is, the fractional derivative and the fractional integral. In 1978, S. Owa [13] gave the following definitions for the fractional calculus.

DEFINITION 1. The fractional integral of order α is defined by

$$D_z^{-\alpha}f(z)=\frac{1}{\Gamma(\alpha)}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\alpha}},$$

where $\alpha>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0} D_z^{-\alpha}f(z).$$

DEFINITION 2. The fractional derivative of order α is defined by

$$D_z^{\alpha}f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z\frac{f(\zeta)d\zeta}{(z-\zeta)^{\alpha}},$$

where $0<\alpha<1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0} D_z^{\alpha}f(z).$$

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha}f(z)=\frac{d^n}{dz^n}D_z^{\alpha}f(z),$$

where $0<\alpha<1$ and $n\in\mathcal{N}\cup\{0\}$.

For other definitions of the fractional calculus, see R. P. Agarwal [1], W. A. Al-Salam [3], T. J. Osler [9], B. Ross [15], K. Nishimoto [8] and M. Saigo [16].

THEOREM 17. *Let a function*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ p\in\mathcal{N})$$

be in the class $\mathcal{T}_p(a,b)$. *Then we have*

$$|D_z^{-\alpha}f(z)|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha}\left\{1-\frac{2b(1-a)p}{(p+1)(1+b)}|z|\right\}$$

and

$$|D_z^{-\alpha}f(z)|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha}\left\{1+\frac{2b(1-a)p}{(p+1)(1+b)}|z|\right\}$$

for $0<\alpha<1$ *and* $z\in\mathcal{U}$. *Further*

$$|D_z^{1-\alpha}f(z)|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha-1}\left\{(p-\alpha)-\frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)}|z|\right\}$$

and

$$|D_z^{1-\alpha}f(z)|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha-1}\left\{(p+\alpha)+\frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)}|z|\right\}$$

for $0<\alpha<1$ *and* $z\in\mathcal{U}$.

Proof. We consider the function

$$\begin{aligned}F(z)&=\frac{\Gamma(p+1+\alpha)}{\Gamma(p+1)}z^{-\alpha}D_z^{-\alpha}f(z)\\&=z^p-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1+\alpha)}{\Gamma(p+n+1+\alpha)\Gamma(p+1)}a_{p+n}z^{p+n}\\&=z^p-\sum_{n=1}^{\infty}A_{p+n}z^{p+n},\end{aligned}$$

where

$$A_{p+n}=\frac{\Gamma(p+n+1)\Gamma(p+1+\alpha)}{\Gamma(p+n+1+\alpha)\Gamma(p+1)}a_{p+n}.$$

By using $0<A_{p+n}<a_{p+n}$ and Theorem 1, we have

$$\begin{aligned}\sum_{n=1}^{\infty}(p+n)(1+b)A_{p+n}&\leqq\sum_{n=1}^{\infty}(p+n)(1+b)a_{p+n}\\&\leqq 2b(1-a)p\end{aligned}$$

which implies that $F(z)$ is in the class $\mathcal{T}_p(a,b)$. Consequently, by Theorem 6, we get

$$|F(z)|\geqq|z|^p-\frac{2b(1-a)p}{(p+1)(1+b)}|z|^{p+1}$$

and

$$|F(z)|\leqq|z|^p+\frac{2b(1-a)p}{(p+1)(1+b)}|z|^{p+1}$$

which give the first half of the theorem. Moreover, by using the second half of Theorem 6,

$$|F'(z)|\geqq p|z|^{p-1}-\frac{2b(1-a)p}{1+b}|z|^p$$

and

$$|F'(z)|\leqq p|z|^{p-1}+\frac{2b(1-a)p}{1+b}|z|^p$$

for $z\in\mathcal{U}$. Hence we have

$$\begin{aligned}|D_z^{1-\alpha}f(z)|&\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{\alpha}\left\{p|z|^{p-1}-\frac{2b(1-a)p}{1+b}|z|^p\right\}\\&\quad-\alpha|z|^{-1}|D_z^{-\alpha}f(z)|\\&\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha-1}\left\{(p-\alpha)-\frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)}|z|\right\}\end{aligned}$$

and

$$\begin{aligned}|D_z^{1-\alpha}f(z)|&\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{\alpha}\left\{p|z|^{p-1}+\frac{2b(1-a)p}{1+b}|z|^p\right\}\\&\quad+\alpha|z|^{-1}|D_z^{-\alpha}f(z)|\\&\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha-1}\left\{(p+\alpha)+\frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)}|z|\right\}\end{aligned}$$

for $0<\alpha<1$ and $z\in\mathcal{U}$.

REMARK 1. If we let $\alpha\to 0$ in Theorem 17, then we have Theorem 6.

COROLLARY 7. *Under the hypotheses of* Theorem 17, $D_z^{-\alpha}f(z)$ *is included in the disk with center at the origin and radius*

$$\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}\left\{1+\frac{2b(1-a)p}{(p+1)(1+b)}\right\}.$$

Further $D_z^{1-\alpha}f(z)$ *is included in the disk with center at the origin and radius*

$$\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}\left\{(p+a)+\frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)}\right.$$

THEOREM 18. *Let a function*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ \ p\in\mathcal{N})$$

be in the class $\mathcal{C}_p(a,b)$. *Then we have*

$$|D_z^{-\alpha}f(z)|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha}\left\{1-\frac{2b(1-a)p^2}{(p+1)^2(1+\mathrm{b})}|z|\right\}$$

and

$$|D_z^{-\alpha}f(z)|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha}\left\{1+\frac{2b(1-a)p^2}{(p+1)^2(1+b)}|z|\right\}$$

for $0<\alpha<1$ *and* $z\in\mathcal{U}$. *Furthermore*

$$|D_z^{1-\alpha}f(z)|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha-1}\left\{(p-\alpha)-\frac{2b(1-a)p^2(p+1+\alpha)}{(p+1)^2(1+b)}|z|\right\}$$

and

$$|D_z^{1-\alpha}f(z)|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}|z|^{p+\alpha-1}\left\{(p+\alpha)+\frac{2b(1-a)p^2(p+1+\alpha)}{(p+1)^2(1+b)}|z|\right\}$$

for $0<\alpha<1$ *and* $z\in\mathcal{U}$.

The proof of Theorem 18 is given in much the same way as Theorem 17 with the aid of Theorem 7.

REMARK 2. If we let $\alpha\to 0$ in Theorem 18, then we have Theorem 7.

COROLLARY 8. *Under the hypotheses of* Theorem 18, $D_z^{-\alpha}f(z)$ *is included in the disk with center at the origin and radius*

$$\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}\left\{1+\frac{2b(1-a)p^2}{(p+1)^2(1+b)}\right..$$

Further $D_z^{1-\alpha}f(z)$ *is included in the disk with center at the origin and radius*

$$\frac{\Gamma(p+1)}{\Gamma(p+1+\alpha)}\left\{(p+\alpha)+\frac{2b(1-a)p^2(p+1+\alpha)}{(p+1)^2(1+b)}\right\}.$$

THEOREM 19. *Let a function*

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (a_{p+n}\geqq 0,\ p\in\mathcal{N})$$

be in the class $\mathcal{C}_p(a,b)$. *Then we have*

$$|D_z^{\alpha}f(z)|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}|z|^{p-\alpha}\left\{1-\frac{2b(1-a)p}{(p+1)(1+b)}|z|\right\}$$

and

$$|D_z^{\alpha}f(z)|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}|z|^{p-\alpha}\left\{1+\frac{2b(1-a)p}{(p+1)(1+b)}|z|\right\}$$

for $0<\alpha<1$ *and* $z\in\mathcal{U}$. *Furthermore*

$$|D_z^{1+\alpha}f(z)|\geqq\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}|z|^{p-1-\alpha}\left\{(p-\alpha)-\frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)}|z|\right\}$$

and

$$|D_z^{1+\alpha}f(z)|\leqq\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}|z|^{p-1-\alpha}\left\{(p+\alpha)+\frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)}|z|\right\}$$

for $0<\alpha<1$ *and* $z\in E$, *where* $E=\mathcal{U}$ *if* $p\geqq 2$ *and* $E=\mathcal{U}-\{0\}$ *if* $p=1$.

Proof. We consider the function

$$\begin{aligned}G(z)&=\frac{\Gamma(p+1-\alpha)}{\Gamma(p+1)}z^{\alpha}D_z^{\alpha}f(z)\\&=z^p-\sum_{n=1}^{\infty}\frac{\Gamma(p+n+1)\Gamma(p+1-\alpha)}{\Gamma(p+n+1-\alpha)\Gamma(p+1)}a_{p+n}z^{p+n}\\&=z^p-\sum_{n=1}^{\infty}B_{p+n}z^{p+n},\end{aligned}$$

where

$$B_{p+n}=\frac{\Gamma(p+n+1)\Gamma(p+1-\alpha)}{\Gamma(p+n+1-\alpha)\Gamma(p+1)}a_{p+n}.$$

Since

$$B_{p+n}\leqq\left(\frac{p+n}{p}\right)a_{p+n}$$

and $f(z)\in\mathcal{C}_p(a,b)$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}(p+n)(1+b)B_{p+n}&\leqq\sum_{n=1}^{\infty}\frac{(p+n)^2}{p}(1+b)a_{p+n}\\&\leqq 2b(1-a)p.\end{aligned}$$

Hence we have $G(z)\in\mathcal{T}_p(a,b)$ by Theorem 1, further by using Theorem 6,

$$|G(z)|\geqq|z|^p-\frac{2b(1-a)p}{(p+1)(1+b)}|z|^{p+1}$$

and

$$|G(z)| \leqq |z|^p + \frac{2b(1-a)p}{(p+1)(1+b)} |z|^{p+1}$$

which imply the first half of the theorem. Moreover we have

$$|G'(z)| \geqq p|z|^{p-1} - \frac{2b(1-a)p}{1+b} |z|^p$$

and

$$|G'(z)| \leqq p|z|^{p-1} + \frac{2b(1-a)p}{1+b} |z|^p$$

with the aid of the second half of Theorem 6. Therefore, by using the first half of the theorem, we obtain

$$\begin{aligned} |D_z^{1+\alpha} f(z)| &\geqq \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} |z|^{-\alpha} \left\{ p|z|^{p-1} - \frac{2b(1-a)p}{1+b} |z|^p \right\} \\ &\qquad -\alpha |z|^{-1} |D_z^\alpha f(z)| \\ &\geqq \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} |z|^{p-1-\alpha} \left\{ (p-\alpha) - \frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)} |z| \right\} \end{aligned}$$

and

$$\begin{aligned} |D_z^{1+\alpha} f(z)| &\leqq \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} |z|^{-\alpha} \left\{ p|z|^{p-1} + \frac{2b(1-a)p}{1+b} |z|^p \right\} \\ &\qquad +\alpha |z|^{-1} |D_z^\alpha f(z)| \\ &\leqq \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} |z|^{p-1-\alpha} \left\{ (p+\alpha) + \frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)} |z| \right\} \end{aligned}$$

for $0<\alpha<1$ and $z \in E$, where $E=\mathcal{U}$ if $p \geqq 2$ and $E=\mathcal{U}-\{0\}$ if $p=1$.

COROLLARY 9. *Under the hypotheses of* Theorem 19, $D_z^\alpha f(z)$ *is included in the disk with center at the origin and radius*

$$\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} \left\{ 1 + \frac{2b(1-a)p}{(p+1)(1+b)} \right\}.$$

Furthermore, let $p \in \mathcal{N}-\{1\}$, *then under the hypotheses of* Theorem 19 $D_z^{1+\alpha} f(z)$ *is included in the disk with center at the origin and radius*

$$\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} \left\{ (p+\alpha) + \frac{2b(1-a)p(p+1+\alpha)}{(p+1)(1+b)} \right\}.$$

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Department of Mathematics
Kinki University
Osaka, Japan