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ON A COMPARISION THEOREM FOR PSEUDOMETRICS OF RIEMANN SURFACES

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Let S be a complex manifold and let ρ be a non-negative real valued function on $S\times S$. If $\rho(x, y)=\rho(y, x)$, $\rho(x, y)+\rho(y, z)\geq\rho(x, z)$ and $\rho(x, x)=0$ for any three points $\{x, y, z\}$ in S, then ρ is called a ***pseudometric*** on S. We call any system which assigns a pseudometric to each complex manifold a *Schwarz-Pick system* if it satisfies the following conditions;

(a) The distance assigned to the unit open disk in the complex plane is the Poincare metric (with constant negatice curvature-4).

(b) If ρ_1 and ρ_2 are pseudometrics assigned to the complex manifolds S_1 and S_2 respectively, then $\rho(h(x), h(y))\leq\rho(x, y)$ for all holomorphic mappings $h: S_1\to S_2$ and for each pair of points x and y in S_1. It is well known that the Kobayashi pseudometric is the largest and the Caratheodory pseudometric is the smallest one which can be assigned to complex manifolds by a Schwarz-Pick system.

The *Kobayashi pseudometric* K on S is defined as follows; For any pair of points x and y on S, we take a finite number of points $x=x_0$, $x_1\ldots$, x_{k-1}, $x_k=y$ of S, points $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$, of the unit disk D in the complex plane, and holomorphic mappings $f_1, \ldots, f_k$ of D into S such that $f(a_i)=x_{i-1}$ and $f(b_i)=x_i$ for $i=1, \ldots, k$. For each choice of points and holomorphic mappings, we consider the number

$$p(a_1, b_1)+\ldots+p(a_k, b_k).$$

Let $K(x, y)$ be the infimum of the numbers obtained in this manner for all possible choices. The function K is called the Kobayashi pseudometric on S. For the definitions and other relevant results about pseudometrics, one can refer to [1] or [2].

The harmonic distance is defined in [4]. For convenience of readers we give the definition in the following. We call a real valued function u on a complex manifold S harmonic (=pluriharmonic) if it is given locally by the real part of a holomorphic function. That is, for each $z_0\in S$, there

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is a holomorphic function defined on an open neighborhood V of z_0 with $\mathrm{real}(f(z))=u(z)$ for all z in V. Let G be the family of all harmonic functions h on the unit disk D which satisfies $h(0)=0$ and $-1<h(z)<1$ for all $z\in D$. Let $m(0,x)=\sup\{h(x) : h\in G\}$, where $0<x<1$. Then it is well known that $m(0,x)=\frac{4}{\pi}\tan^{-1}x$. Let $\mathrm{m}(z_1,z_2)=\sup\{h(z_1),h(z_2) : h\in F\}$, where F denotes all of the harmonic functions $h: S\to(-1,1)$ with $h(z_1)=0$ or $h(z_2)=0$. Set $n(z_1,z_2)=\tan\frac{\pi}{4}m(z_1,z_2)$. The *harmoni distance* H is defined by the real valued function $H: S\times S\to R$, with $H(z_1,z_2)=P(0, n(z_1,z_2))$, where P is the Poincare metric of the unit disk D. It is clear that H satisfies Schwarz-Pick system except triangle inequality.

MAIN RESULTS: In the above we defined $m(0,x)=\left\{\frac{4}{\pi}\tan^{-1}x,\right.$ for positive real x in $\left.D\right\}$, we need a description of the harmonic function $H: D\to(-1,1)$ with $h(x)=m(0,x)$. The function h is well known, but, for convenience of our discussion, we describe it in the following. Consider the holomorphic one to one map f of the unit disk D onto $I=\{z : |\mathrm{real}(z)|<1\}$ with the conditions $f(0)=0$ and $f(x)\geq 0$ for $0\leq x<1$. Then $f(z)$ is given by

$$\frac{2i}{\pi}\operatorname{Log}\frac{z+i}{1+iz}+1,\ \text{where } i=\sqrt{-1}.$$

Let $z=re^{i\theta}$. Then the argument of $z+i/1+iz$ is given

$$\frac{\pi i}{2}-i\tan^{-1}\frac{2r\cos\theta}{1-r^2}.$$

Hence we have

$$\mathrm{real}(f(z))=\frac{2}{\pi}\ \frac{2r\cos\theta}{1-r^2}$$

and

$$f(x)=\frac{4}{\pi}\tan^{-1}x.$$

For convenience we describe this function in the following lemma.

LEMMA 1. *Let f be the holomorphic one to one map of the unit disk $D=\{z : |z|<1\}$ onto $I=\{z : -1<\mathrm{real}(z)<1\}$. With the restriction $f(0)=0$ and $f(x)\geq 0$ for all $0\leq x<1$ and $x\in D$. Then it is given by*

$$f(z)=\frac{2i}{\pi}\operatorname{Log}\frac{z+i}{1+iz}+1,\ \text{where } \sqrt{-1}=i.$$

The real part of $f(z)$ gives the maximal harmonic function

$$real(f(z)) = \frac{2}{\pi}\tan^{-1}\frac{2r\cos\theta}{1-r^2}, \quad where\ z = re^{i\theta},$$

and

$$f(x) = m(0, x) = \frac{4}{\pi}\tan^{-1}x.$$

Let G be the family of all harmonic functions $h : S \to (-1, 1)$ $h(z_1)=0$ or $h(z_2)=0$. Set $m(z_1, z_2) = \sup\{h(z_1), h(z_2) : h \in G\}$. The function space G is compact for the uniform convergence on the compact subsets of S. Hence there is an element g in G with $g(z_1) = m(z_1, z_2)$ or $g(z_2) = m(z_1, z_2)$, we call such a function the *maximal harmonic* function with respect to z_1 and z_2.

LEMMA 2. *Let S be a Riemann surface which admits the unit disk D as the holomorphic covering surface. Let h be the maximal harmonic function with respact to z_1 and z_2 with $h(z_1)=0$. Then $K(z_1, z_2) = H(z_1, z_2)$ holds for any distinct points in S if and only if there is a covering map $\pi : D \to S$ with $\pi(0) = z_1$, $\pi(x_1) = z_2$, $K(0, x) = K(z_1, z_2)$ and $\pi \circ h = \frac{2}{\pi}\tan^{-1}\frac{2r\cos\theta}{1-r^2}$.*

Proof. Suppose $\pi \circ h = \frac{2}{\pi}\tan^{-1}\frac{2r\cos\theta}{1-r^2}$ holds with the covering map π of the above. Then it is clear by the definition of the pseudometric H, we have $H(z_1, z_2) = K(z_1, z_2)$.

It is known that we can choose the covering map π so that it satisfies $\pi(0) = z_1$, $\pi(x) = z_2$ and $K(0, x) = K(z_1, z_2)$ (see [2]). Let h be the maximal harmonic function with respect to z_1 and z_2 with $h(z_1)=0$. Then $\pi \circ h$ is a harmonic function with $\pi \circ h(0) = 0$ and $\pi \circ h(x) = m(z_1, z_2) \geq 0$. Suppose $K(z_1, z_2) = H(z_1, z_2)$ is true. Then we have $K(0, x) = K(z_1, z_2) = H(z_1, z_2) = H(0, x)$. Hence we have

$$m(z_1, z_2) = m(0, x) = \frac{2}{\pi}\tan^{-1}\frac{2x}{1-x^2} = \frac{4}{\pi}\tan^{-1}x.$$

The harmonic function $\pi \circ h$ has values

$$\pi \circ h(0) = 0 \text{ and } \pi \circ h(x) = \frac{4}{\pi}\tan^{-1}x.$$

Now using the Schwarz lemma we conclude that $\pi \circ h$ is the real part of $\left(\frac{2i}{\pi}\operatorname{Log}\frac{z+i}{z+iz}+1\right)$. Hence we proved that

$$\pi \circ h(re^{i\theta}) = \frac{2i}{\pi}\tan^{-1}\frac{2r\cos\theta}{1-r^2}.$$

Now we prove the main theorem. A complex manifold S is called *hyperbolic* if the Kobayahi pseudometric is a metric on S.

THEOREM. *Let S be a hyperbolic Riemann surface which is not the unit disk and an annulus. Then the Kobayashi metric is strictly bigger than the Harmonic distance.*

Proof. Let S be a hyperbolic surface (the unit disk is the holomorphic covering space), and assume $K(z_1, z_2)=H(z_1, z_2)$ for a pair of distinct points z_1 and z_2 in S. Then by the above lemma we can choose a holomorphic covering map $\pi : D\rightarrow S$ and a harmonic function $h : S\rightarrow(-1, 1)$, such that

$$\pi\circ h(re^{i\theta})=\frac{2}{\pi}\tan^{-1}\frac{2r\cos\theta}{1-r^2}.$$

Let G be the holomorphic covering transformation group of the covering π. Then we know that $\pi\circ h(A(z))=\pi\circ h(z)$ holds for all A in G and $z\in D$. For the proof of the theorem it suffices to show that only an annulas admits such a group. For the calculation of the group G, we choose $I=\{z : Im(z) >0\}$ as the covering space. Let $w(z)=\frac{i-z}{iz-1}$; w is a one to one holomorphic map of I onto D.
Let

$$h(z)=\frac{2}{\pi}\tan^{-1}\frac{2r\cos\theta}{1-r^2}=\frac{2}{\pi}\tan^{-1}\frac{z+\bar{z}}{1-z\bar{z}},$$

then $w\circ h(z)$ can be represented as

$$\frac{2}{\pi}\tan^{-1}\frac{z+\bar{z}}{z-\bar{z}}.$$

Now any element $A\in G$ has the form $\frac{az+b}{cz+d}$, $ad-bc=1$ and a, b, c, d are real. Consider the relation $w\circ h(A(z))=w\circ h(z)$. Then by a simple calculation we have

$$\frac{A(z)}{z}=\left(\frac{\overline{A(z)}}{z}\right).$$

It follows that $A(z)=rz$ for some real number r. By the above discussion we know that G is trivial or it is generated by an element of the form $A(z)=rz$. Hence we conclude that S must be an annulus or the unit disk, and proved the theorem. In the above we used the fact that G is a Fuchsian group, for this one can consult with [3] or [5].

COROLLARY. *Let S be a Riemann surface. Then we have the following relations between K and H;*

(i) *If S is not hyperbolic, then $K(z_1, z_2)=H(z_1, z_2)=0$ for all z_1 and z_2*

in S.

(ii) *If S is the unit disk, then* $K(z_1, z_2)=H(z_1, z_2)$ *and* $H(z_1, z_2)>0$ *for* $z_1 \neq z_2$ *in S.*

(iii) *If S is an annulus, then* $K(z_1, z_2) \geq H(z_1, z_2)$ *and the inequality is true for some pair in S.*

(iv) *If S is a Riemann surface not in the above (i)~(iv), then* $K(z_1, z_2) > H(z_1, z_2)$ *for every* z_1 *and* z_2 $(z_1 \neq z_2)$ *in S.*

References

1. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Marcel Dekker N.Y., (1970).
2. S. Kobayashi, *Intrinsic distances measure and geometric function theory*, Bull. Amer. Math. Soc. **82** (1976), 357-456.
3. I. Kra, *Automorphic forms and Kleinian groups*, Reading Massachusets, Addision Wesely, (1972).
4. S.M. Kim, *Harmonic pseudometric and the product property of pseudometrics*, Kyungpook, Math. J. **20** (1980), 239-243.
5. J. Lehner, *Discontinuous groups and automorphic functions*, Amer. Math. Soc. (1964).

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