J. Korean Math. Soc.
Vol. 20, No. 2, 1983

ON ANTI-INVARIANT SUBMANIFOLDS OF COSYMPLECTIC MANIFOLDS

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0. Introduction

A normal almost contact metric manifold is said to be cosymplectic if its fundamental 2-form and contact form are both closed. Cosymplectic manifolds and their submanifolds have been studied by D. E. Blair ([1], [2]), G. D. Ludden ([2], [11]), S. I. Goldberg ([8]), S. S. Eum ([4], [5], [17]). and U-H. Ki ([9], [17]) and others.

In the last decade, the study of anti-invariant submanifolds of Kaehlerian and Sasakian manifolds has provided us with a great deal of new and valuable results ([3], [10], [12], [16], [18], etc.). However, the study of anti-invariant submanifolds of cosymplectic manifolds is not performed yet.

The purpose of the present thesis is to study anti-invariant submanifolds of cosymplectic manifolds and obtain some results. We classify anti-invariant submanifolds of cosymplectic manifolds into two parts. The first part is tangential anti-invariant submanifolds and the second part is normal anti-invariant submanifolds.

In chapter I, we recall fundametal concepts of cosymplectic manifolds and prepare structure equations for anti-invariant submanifolds of cosymplectic manifolds. Lastly we obtain some propositions.

In chapter II, we study anti-invariant submanifolds, which are tangent to the structure vector field, of cosymplectic manifolds. We obtain some basic formulas and define η-umbilical submanifolds of cosymplectic manifolds. We investigate anti-invariant submanifolds of cosymplectic manifolds of constant curvature with respect to r_{ji}. We also study anti-invariant submanifolds with parallel f-structure in the normal bundle and anti-invariant submanifolds of cosymplectic manifolds with vanishing cosymplectic Bochner curvature tensor.

In chapter III, we study anti-invariant submanifolds, which are normal to the structure vector field, of cosymlpectic manifolds. We obtain some basic formulas and investigate the Ricci tensor and scalar curvature of normal

Received November 3. 1982

anti-invariant submanifolds of $M(k)$, where $M(k)$ denotes a cosymplectic manifold of constant ϕ-holomorphic sectional curvature k. We also study anti-invariant submanifolds with parallel f-structure and a normal anti-invariant submanifold of a cosymplectic manifold with vanishing cosymplectic Bochner curvature tensor.

I. Cosymplectic manifolds and submanifolds of cosymplectic manifolds

1. Cosymplectic manifolds

Let M be a $(2m+1)$-dimensional differentiable manfiold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^h\}$ in which there are given a tensor field ϕ of type $(1,1)$, a vector field ξ^h and a 1-form η_h satisfying

$$(1.1)\qquad \phi_j{}^i\phi_i{}^h=-\delta_j^h+\eta_j\xi^h, \qquad \phi_j{}^h\xi^j=0,$$
$$\eta_i\phi_j{}^i=0, \qquad \eta_i\xi^i=1,$$

where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots\cdots, 2m+1\}$. Such a set of a tensor field of type $(1,1)$, a vector field and a 1-form is called *almost contact tructure* and a manifold with an almost contact structure *an almost contact manifold.*

If, in an almost contact manifold, there is given a Riemannian metric g_{ji} such that

$$(1.2)\qquad g_{ts}\phi_j{}^t\phi_i{}^s=g_{ji}-\eta_j\eta_i, \qquad \eta_i=g_{ij}\xi^j,$$

then the manifold is called *an almost contact metric manifold.*

If we put $\phi_{ji}=\phi_j{}^t g_{ti}$, we see from (1.1) and (1.2) that ϕ_{ji} is skew-symmetric. By means of the second relationship of (1.2), we shall write η^h instead of ξ^h in the sequel.

The almost contact structure is said to be *normal* if

$$[\phi, \phi]+d\eta\otimes\xi=0,$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor formed with ϕ and d the operator of the exterior derivative.

A normal almost contact metric structure is said to be *cosymplectic* if the 2-form ϕ_{ji} and the 1-form η_i are both closed. It is known in [1] that the cosymplectic structure is characterized by

$$(1.3)\qquad \nabla_k\phi_j{}^i=0 \text{ and } \nabla_k\eta^i=0,$$

where ∇_k denotes the operator of covariant differentiation with respect to g_{ji}.

If we denote the curvature tensor, Ricci tensor and scalar curvature of a cosymplectic manifold M by $K_{kji}{}^h$, K_{ji} and K respectively, then the *cosymplectic Bochner curvature tensor* ([4]) is defined by

$$B_{kji}{}^h = K_{kji}{}^h + (\delta_k^h - \eta_k \eta^h) L_{ji} - (\delta_j^h - \eta_j \eta^h) L_{ki} + L_k{}^h (g_{ji} - \eta_j \eta_i) - L_j{}^h (g_{ki} - \eta_k \eta_i) + \phi_k{}^h M_{ji} - \phi_j{}^h M_{ki} + M_k{}^h \phi_{ji} - M_j{}^h \phi_{ki} - 2(M_{kj} \phi_i{}^h + \phi_{kj} M_i{}^h), \tag{1.4}$$

where

$$L_{ji} = -\frac{1}{2(m+2)} \{K_{ji} + L(g_{ji} - \eta_j \eta_i)\},\ L_j{}^h = L_{jt} g^{th}, \tag{1.5}$$

$$M_{ji} = -L_{jt} \phi_i{}^t,\ M_j{}^h = M_{jt} g^{th},\ L = L_{ji} g^{ji} = \frac{-1}{4(m+1)} K. \tag{1.6}$$

We recall here the following lemma.

LEMMA. ([7]). *The cosympletic Bochner curvature tensor in a cosymplectic manifold M satisfies the following equation*

$$\nabla_t B_{kji}{}^t = -2m \Big[\nabla_k L_{ji} - \nabla_j L_{ki} + \frac{1}{8(m+1)(m+2)} (\phi_k{}^t \phi_{ji} - \phi_j{}^t \phi_{ki} - 2\phi_i{}^t \phi_{kj}) \nabla_t K\Big].$$

In a cosymplectic manifold M, we call a sectional curvature

$$k = -\frac{g(K(\phi X, X)\phi X, X)}{g(X, X) g(\phi X, \phi X)}$$

determined by two orthogonal vectors X and ϕX the ϕ-holomorphic sectional curvature with respect to the vector X of M. If the ϕ-holomorphic sectional curvature is always constant with respect to any vector at every point of the manifold M, then we call the manifold M *a manifold of constant ϕ-holomorphic sectional curvature.* If a cosymplectic manifold has a constant ϕ-holomorphic sectional curvature k at every point, then the components of the curvature tensor of the manifold are of the form ([5], [11])

$$K_{ijkl} = -\frac{k}{4} (g_{ik} g_{jl} - g_{jk} g_{il} + \phi_{ik} \phi_{jl} - \phi_{jk} \phi_{il} + 2\phi_{ij} \phi_{kl} - \eta_i \eta_k g_{jl} + \eta_j \eta_k g_{il} + \eta_i \eta_l g_{jk} - \eta_j \eta_l g_{ik}). \tag{1.7}$$

A cosymplectic manifold M is said to be of constant curvature space with respect to γ_{ji} if the curvature tensor is of the form

$$K_{kji}{}^t = c(\gamma_{ji} \gamma_k{}^t - \gamma_{ki} \gamma_j{}^t), \tag{1.8}$$

where $\gamma_{kj} = g_{kj} - \eta_k \eta_j$ ([6]).

2. Submanifolds of cosymplectic manifolds

Let M be a $(2m+1)$-dimensional cosymplectic manifold with structure tensors $(\phi_j^h,\ g_{ji},\ \eta^h)$, and let N be an n-dimensional Riemanian manifold covered by a system of coordinate neighborhoods $\{V;\ y^a\}$ and isometrically immersed in M by the immersion $i: N\to M$. Suppose that the local expression of the submanifold N in M is

(2.1) $$x^h=x^h(y^a).$$

Here and in the sequel the indices a, b, c, d, e run over the range $\{1, 2, 3, \cdots, n\}$.

Differentiate (2.1) and put

(2.2) $$B_b^h=\partial_b x^h, \qquad \partial_b=\partial/\partial y^b,$$

which is, for each fixed index b, a local vector field tangent to N. These local vector fields B_b^h span the tangent plane of N at each point of N. Denote by C_x^h $2m+1-n$ mutually orthogonal unit normals to N. The indices x, y, z run over the range $\{n+1, \cdots\cdots, 2m+1\}$. If we denote by g_{bc} the metric tensor on N induced from that of M, then we have $g_{bc}=g_{ji}B_b^jB_c^i$.

The transforms of B_c^j by ϕ can be expressed as linear combinations of B_a^h and C_x^h, that is,

(2.3) $$\phi_j^hB_c^j=f_c^aB_a^h-f_c^xC_x^h,$$

where f_c^a is a tensor field of type (1, 1) defined on N, f_c^x a 1-form for each fixed index x. The transforms ϕC_x^j of C_x^j by ϕ can be expressed as

(2.4) $$\phi_j^hC_x^j=f_x^aB_a^h+q_x^yC_y^h,$$

where f_x^c is vector field of N and q_x^y is a function for each fixed x and y.

If we denote by ∇_c the operator of the van der Waerden-Bortolotti covariant differentiation with respect to the Christoffel symbols $\left\{ {a \atop b\,c} \right\}$ formed with g_{bc}, we have the equations of Gauss and Weingarten for N

(2.5) $$\nabla_cB_b^h=h_{cb}{}^xC_x^h,$$

(2.6) $$\nabla_cC_x^h=-h_{cx}{}^aB_a^h$$

respectively, where $h_{cb}{}^x$ is the second fundamental tensor with respect to the normal C_x^h and $h_c{}^a{}_x=h_{cb}{}^yg^{ba}g_{yx}$, g_{yx} being the metric tensor of the normal bundle of N given by $g_{yx}=g_{ji}C_y^jC_x^i=\delta_{yx}$.

The structure equations of the submanifold N are given by

(2.7) $$R_{abcd}=K_{ijkl}B_a^iB_b^jB_c^kB_d^l+h_{adx}h_{bc}{}^x-h_{bdx}h_{ac}{}^x,$$

(2.8) $$K_{ijk}{}^lB_b^iB_c^jB_a^kC^x{}_l=\nabla_bh_{ca}{}^x-\nabla_ch_{ba}{}^x,$$

(2.9) $$K_{ijk}{}^lB_b^iB_c^jC_x^kC^y{}_l=R_{bcx}{}^y-(h_{be}{}^yh_c{}^e{}_x-h_{ce}{}^yh_b{}^e{}_x),$$

where R_{abcd} is the curvature tensor of N and $R_{dcy}{}^x$ is the curvature tensor of the connection induced in the normal bundle. The above equations (2.7), (2.8) and (2.9) are the equations of Gauss, Codazzi and Ricci respectively.

The structure vector field η^h can be written as

$$\eta^h=\eta^a B_a{}^h+\eta^x C_x{}^h. \tag{2.10}$$

Applying the operator ϕ to both sides of (2.3), (2.4) and (2.10), using (1.1), (1.2) and comparing tangential parts and normal parts of both sides respectively, we obtain

(2.11) $f_c{}^a f_a{}^b=-\delta_c^b+\eta_c\eta^b+f_c{}^x f_x{}^b$,

(2.12) $f_c{}^a f_a{}^y=-f_c{}^x q_x{}^y-\eta_c\eta^y$,

(2.13) $q_x{}^y q_y{}^z=-\delta_x^z+\eta_x\eta^z+f_x{}^a f_a{}^z$,

(2.14) $q_x{}^y f_y{}^b=\eta_x\eta^b-f_x{}^a f_a{}^b$,

(2.15) $f_a{}^b\eta^a+f_x{}^b\eta^x=0$,

(2.16) $f_a{}^x\eta^a-q_z{}^x\eta^z=0$,

(2.17) $f_{cd}+f_{dc}=0$,

(2.18) $q_{xy}+q_{yx}=0$,

where $q_{xy}=q_x{}^z g_{zy}$ and $f_{ab}=f_a{}^c g_{cb}$.

Transvecting η_h to both sides of (2.3), (2.4) and (2.10), using (1.1), (1.2) and comparing tangential parts and normal parts of both sides respectively, we find

(2.19) $f_{ca}\eta^a-f_c{}^x\eta_x=0$,

(2.20) $q_x{}^y\eta_y+f_x{}^a\eta_a=0$,

(2.21) $f_{cy}=f_{yc}$,

(2.22) $\eta^a\eta_a+\eta_x\eta^x=1$,

where $f_{cy}=f_c{}^z g_{zy}$ and $f_{yc}=f_y{}^a g_{ac}$.

A submanifold N immersed in a cosymplectic manifold M is said to be anti-invariant in M if $\phi(T_x(N))\subset T_x(N)^\perp$ for each point x in N, where $T_x(N)$ and $T_x(N)^\perp$ denote the tangent and normal space of N at x respectively. In an anti-invariant submanifold N we have $f_c{}^a=0$ because of (2.3).

Contracting (2.11) with respect to c and b, we have

$$-f_{ab}f^{ab}=-n+\eta_c\eta^c+f_a{}^x f_x{}^a. \tag{2.23}$$

Contracting with respect to x and z in equation (2.13), we obtain

$$f_a{}^x f_x{}^a=2m+1-n-\eta_x\eta^x-q_{xy}q^{xy}. \tag{2.24}$$

Substituting (2.24) into (2.23), we obtain

$$f_{ab}f^{ab}=q_{xy}q^{xy}-2(m-n)-1+\eta_x\eta^x-\eta_c\eta^c. \tag{2.25}$$

Thus we have the following two propositions with the aid of (2.22) and (2.25)

PROPOSITION 1([16]). *Let M be a cosymplectic manifold of dimension $2m+1$ and let N be an $(m+1)$-dimensional anti-invariant submanifold of M. Then the structure vector field η^h is tangent to N and $q_{xy}=0$.*

PROPOSITION 2. *Let M be a cosymplectic manifold of dimension $2m+1$ and let N be an m-dimensional anti-invariant submanifold of M. Then we have $q_{xy}q^{xy}=2\eta_c\eta^c$.*

Moreover, if the structure vector field η^h is normal to N, them we have

$q_{xy}=0$.

Suppose that N is a submanifold of the cosymplectic manifold M of constant ϕ-holomorphic sectional curvature k. Then we have from (1.7)

$$K_{ijkl}C_x^iB_a^jB_b^kB_c^l = -\frac{k}{4}(f_{xb}f_{ac}-f_{xc}f_{ab}+2f_{xa}f_{bc}-\eta_x\eta_bg_{ac}+\eta_x\eta_cg_{ab}). \tag{2.26}$$

If we apply the operator ∇_b of the covariant differentiation to (2.3), (2.4) and (2.10) respectively and take account of (1.3), (2.5) and (2.6), then we obtain

(2.27) $\nabla_bf_c^a=h_{bc}^xf_x^a-f_c^xh_b^a{}_x$, (2.28) $\nabla_bf_c^y=-h_{bc}^xq_x^y+f_c^eh_{be}^y$,
(2.29) $\nabla_bq_x^z=h_b^e{}_xf_e^z-f_x^eh_{be}^z$, (2.30) $\nabla_b\eta^a=h_b^a{}_x\eta^x$,
(2.31) $\nabla_b\eta_x=-\eta^eh_{ebx}$.

II. Tangential anti-invariant submanifolds of cosymplectic manifolds

In this chapter we investigate anti-invariant submanifolds, which are tangent to the structure vector field, of cosymplectic manifolds. We call such a submanifold tangential anti-invariant submanifold.

1. Basic formulas of tangential anti-invariant submanifolds

Let N be a n-dimensional tangential anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $M(n\leqq m+1)$. Then we have $f_c^a=0$ and $\eta^x=0$.
Therefore the equations of chapter I which contain f_c^a or η^x can be rewritten as

(1.1) $\eta^h=\eta^aB_a^h$, (1.2) $-\delta_c^b+\eta_c\eta^b+f_c^xf_x^b=0$,
(1.3) $f_c^xq_x^y=0$, (1.4) $q_x^yq_y^z=-\delta_x^z+f_x^af_a^z$,
(1.5) $f_a^x\eta^a=0$, (1.6) $\eta_a\eta^a=1$,
(1.7) $f_a^xf_x^a=n-1$, (1.8) $q_{xy}q^{xy}=2(m-n)+2$,
(1.9) $\eta^eh_{eb}^x=0$, (1.10) $\nabla_b\eta^a=0$.

If we transvect the equation (1.4) with q_w^x and take account of (1.3), then we have

$$q_w^xq_x^yq_y^z+q_w^z=0.$$

If q_x^y does not vanish on N, then it defines an f-structure in the normal bundle ([14]). If $\nabla_bq_x^z=0$, then the f-structure in the normal bundle is said to be *parallel*.

We now consider the system of partial differential equations in a cosymplectic manifold M of constant curvature c with respect to γ_{ji}

(1.11) $$\nabla_j X_i = c\gamma_{ji} + X_j X_i,\ X_i \eta^i = 0$$

for unknown vector X_j.

If we differentiate (1.11) coveriantly and take account of (1.3) of chapter I, then we have

$$\nabla_k \nabla_j X_i = (c\gamma_{kj} + X_k X_j) X_i + (c\gamma_{ki} + X_k X_i) X_j,$$

which implies

$$\nabla_k \nabla_j X_i - \nabla_j \nabla_k X_i = c(\gamma_{ki} X_j - \gamma_{ji} X_k) = c(\gamma_{ki} \gamma_j{}^h - \gamma_{ji} \gamma_k{}^h) X_h.$$

The necessary and sufficient condition for the system (1.11) to be completely integrable is

$$\nabla_k \nabla_j X_i - \nabla_j \nabla_k X_i = -K_{kji}{}^h X_h.$$

Since the curvature tensor of M has the form of (1.8) of chapter I, this condition is satisfied. Hence the system is completely integrable.

Let us consider a tangential anti-invariant submanifold N of M such that N has a solution vector X_j of (1.11) as a normal vector, that is, $X_j = \lambda C_{*j}$ for a function λ. Transvecting (1.11) with $B_b{}^j B_c{}^j$, we obtain

(1.12) $$B_b{}^j B_c{}^j \nabla_j X_j = c(g_{bc} - \eta_b \eta_c).$$

If we substitute $X_i = \lambda C_{*i}$ into (1.12), then we have $\lambda B_b{}^i \nabla_c C_{*i} = c(g_{bc} - \eta_b \eta_c)$. Hence we have $h_{cb}{}^* = \alpha^* (g_{bc} - \eta_b \eta_c)$ for a certain index $_*$, where α^* denotes $-c/\lambda$.

If the second fundamental tensor $h_{ab}{}^x$ of a tangential anti-invariant submanifold of a cosymplectic manifold is of the form

(1.13) $$h_{bc}{}^x = \alpha^x (g_{bc} - \eta_b \eta_c)$$

for each index x, then we call such a submanifold η-umbilical submanifold. Here α^x denotes a function on N for each index x.

2. Tangential anti-invariant submanifolds of constant curvature spaces with respect to γ_{ji}

In this section we consider a η-umbilical submanifold N of a constant curvature space $M(c)$ with respect to γ_{ji}. The equations (2.7) and (1.8) of chapter I and (1.13) imply

(2.1) $$R_{abcd} = (c+\alpha)(g_{bc} g_{ad} - g_{ac} g_{bd} + \eta_a \eta_c g_{bd} - \eta_b \eta_c g_{ad} - \eta_a \eta_d g_{bc} + \eta_b \eta_d g_{ac}),$$

where α denotes $\alpha^x \alpha_x$.

Transvecting the equation (2.1) with g^{ad}, we have

(2.2) $$R_{bc} = (n-2)(c+\alpha)(g_{bc} - \eta_b \eta_c).$$

Transvecting the equation (2.2) with g^{bc}, we have

(2.3) $$R=(n-1)(n-2)(c+\alpha).$$

From (2.8) and (1.8) of chapter I we have

(2.4) $$\nabla_a h_{bc}{}^x-\nabla_b h_{ac}{}^x=0.$$

If we differentiate (1.13) covariantly along N, then we have

(2.5) $$\nabla_a h_{bc}{}^x=\nabla_a \alpha^x(g_{bc}-\eta_b\eta_c),$$

from which we find

(2.6) $$\nabla_a\alpha^x(g_{bc}-\eta_b\eta_c)-(\nabla_b\alpha^x)(g_{ac}-\eta_a\eta_c)=0.$$

Transvecting (2.6) with g^{bc}, we obtain

(2.7) $$(n-2)\nabla_a\alpha^x+\eta_a\eta^e\nabla_e\alpha^x=0.$$

We now assume that $n>2$. If we transvect (2.7) with η^a, we have $\eta^e\nabla_e\alpha^x=0$. Substituting $\eta^e\nabla_e\alpha^x=0$ into (2.7), we find

(2.8) $$\nabla_a\alpha^x=0,$$

from which we have

(2.9) $$\nabla_a\alpha=0.$$

From (1.1), (2.2) and (2.3) we have

(2.10) $$\nabla_a R_{bc}=0, \quad \nabla_a R=0.$$

From (1.10), (2.1) and (2.9), we obtain

$$\nabla_e R_{abcd}=0,$$

which means that the submanifold N is locally symmetric. Thus we have the following theorem.

THEOREM 3. *Let N be an n-dimensional tangential anti-invariant submanifold of cosymplectic manifold M of constant curvature with respect to γ_{ji} ($n>2$). If N is a η-umbilical submanifold of M, then N is locally symmetric.*

The Weyl conformal curvature tensor field of N is the tensor field C of type (1, 3) defined by

(2.11) $$C_{abc}{}^d=R_{abc}{}^d+\frac{1}{n-2}(R_{ac}\delta_b^d-R_{bc}\delta_a^d+g_{ac}R_b{}^d-g_{bc}R_a{}^d)-\frac{R}{(n-1)(n-2)}(g_{ac}\delta_b^d-g_{bc}\delta_a^d).$$

Moreover, we put

(2.12) $$C_{abc}=\nabla_a R_{bc}-\nabla_b R_{ac}-\frac{1}{2(n-1)}(g_{bc}\nabla_a R-g_{ac}\nabla_b R).$$

It is well-known that a necessary and sufficient condition for a Riemannian manifold to be conformally flat is that $C_{abc}{}^d=0$ for $n>3$ and $C_{abc}=0$ for

$n=3$.

Now let us compute the Weyl conformal curvature tensor of a η-umbilical submanifold N of the constant curvature space with respect to γ_{ji}. Substituting (2.1), (2.2) and (2.3) into the equation (2.11), we obtain $C_{abc}{}^{d}=0$ provided that $n>3$. For $n=3$ we have $C_{abc}=0$ by the help of (2.10) and (2.12). Thus we have the following

THEOREM 4. *If N is an n-dimensional η-umbilical submanifold of a constant curvature space M with respect to γ_{ji} $(n\geqq 3)$, then N is conformally flat.*

3. Tangential anti-invariant submanifolds with parallel f-structure in the normal bundle

In section 1 we have shown that $q_x{}^y$ defines an f-structure in the normal bundle of N. In this section we investigate tangential anti-invariant submanifold with parallel f-structure in the normal bundle.

Let N be an n-dimensional tangential anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold and suppose that the f-structure $q_x{}^y$ in the normal bundle ls parallel, that is, $\nabla_b q_x{}^y=0$. Then by the equation (2.29) of chapter I we have

(3.1) $$h_b{}^e{}_x f_e{}^y - f_x{}^e h_{be}{}^y = 0.$$

Transvecting the equation (3.1) with $f_c{}^x$, we have

$$h_b{}^e{}_x f_c{}^x f_e{}^y - f_c{}^x f_x{}^e h_{be}{}^y = 0.$$

If we substitute (1.2) into this equation and take account of (1.9), then we have

(3.2) $$h_{bc}{}^y = h_b{}^e{}_x f_c{}^x f_e{}^y.$$

From (3.2), (1.2) and (1.9), we obtain

(3.3) $$\begin{aligned} h_{ad}{}^x h_{bcx} &= h_a{}^e{}_z f_d{}^z f_e{}^x h_b{}^g{}_y f_c{}^y f_{gx} \\ &= h_a{}^e{}_z f_d{}^z h_b{}^g{}_y f_c{}^y (g_{eg} - \eta_e \eta_g) \\ &= h_a{}^e{}_z h_{eby} f_d{}^z f_c{}^y. \end{aligned}$$

The equation (3.3) implies

(3.4) $$h_{ad}{}^x h_{bcx} - h_b{}^x h_{acx} = (h_a{}^e{}_z h_{eby} - h_b{}^e{}_z h_{eay}) f_d{}^z f_c{}^y.$$

If the submanifold N has commutative second fundamental tensors, that is,

(3.5) $$h_a{}^e{}_x h_{eby} = h_a{}^e{}_y h_{ebx},$$

then the equation (3.4) implies

(3.6) $$h_{ad}{}^x h_{bcx} - h_{bd}{}^x h_{acx} = 0.$$

From the last equation and the equation (2.7) of chapter I we obtain the following

PROPOSITION 5. *Let N be a tangential anti-invariant submanifold of a cosymplectic manifold M. If N has the parallel f-structure $q_x{}^y$ in the normal bundle and has the commutative second fundamental tensors, then we have*

$$R_{abcd}=K_{ijkl}B_a{}^iB_b{}^jB_c{}^kB_d{}^l.$$

Let N be an n-dimensional tangential anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $M(k)$ of constant ϕ-holomorphic sectional curvature k. Then the equations (1. 7) and (2. 7) of chapter I imply

$$R_{abcd}=-\frac{1}{4}k(g_{ac}g_{bd}-g_{bc}g_{ad}-\eta_a\eta_c g_{bd}+\eta_b\eta_c g_{ad}+\eta_a\eta_d g_{bc}-\eta_b\eta_d g_{ac})+h_{ad}{}^x h_{bcx}-h_{bd}{}^x h_{acx} \tag{3.7}$$

where we have used $\phi_{ij}B_a{}^iB_b{}^j=0$.

Suppose that N has the parallel f-structure $q_x{}^y$ in the normal bundle and has the commutative second fundamental tensors. Then (3. 6) and (3. 7) imply

$$R_{abcd}=-\frac{k}{4}(g_{ac}g_{bd}-g_{bc}g_{ad}-\eta_a\eta_c g_{bd}+\eta_b\eta_c g_{ad}+\eta_a\eta_d g_{bc}-\eta_b\eta_d g_{ac}), \tag{3.8}$$

from which we have

$$R_{bc}=\frac{k}{4}(n-2)(g_{bc}-\eta_b\eta_c) \tag{3.9}$$

and

$$R=\frac{k}{4}(n-1)(n-2). \tag{3.10}$$

Substituting (3. 8), (3. 9) and (3. 10) into the equation (2. 11), we obtain $C_{abc}{}^d=0$ and $C_{abc}=0$ by virtue of (1. 10). Thus we have the following

THEOREM 6. *Let N be an n-dimensional tangential anti-invariant submanifold of a cosymplectic manifold M(k) of constant ϕ-holomorphic sectional curvature k. If the f-structure is parallel in the normal bundle and the second fundamental tensors are commutative, then N is conformally flat.*

On the other hand, since η^a is parallel unit vector and N is conformally flat (or the curvature tensor of N has the form (3. 8)) we have, by a theorem of K. Yano ([13]),

THEOREM 7. *Let N be an $(n+1)$-dimensional tangential anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M(k) of constant ϕ-holomorphic sectional curvature k. If the f-structure in the normal bundle is parallel and the second fundamental tensors are commutative, then N is locally a Riemannian direct product $N^n\times R^1$, where R^1 is a straight line and N^n is* *an n-dimensional Riemannian space of constant curvature* $\frac{k}{4}$.

If N is an $(m+1)$-dimensional anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M, then N is a tangential submanifold of M(cf. proposition 1) and $q_x{}^y=0$. Therefore we also have the equation (3.1). From the similar arguments of theorem 7 we have the following

THEOREM 8. *Let N be an $(m+1)$-dimensional anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $M(k)$. If the second fundamental tensors are commutative, then N is locally a Riemannian direct product $N^m \times R^1$, where R^1 is a straight line and N^m is an m-dimensional Riemannian space of constant curvature* $\frac{k}{4}$.

We now consider the normal connection of an n-dimensional tangential anti-invariant submanifold of a cosymplectic manifold $M(k)$. Substituting (1.7) of chapter I into the equation of Ricci (chapter I. (2.9)), we have

$$R_{abxy} = -\frac{k}{4}(f_{ax}f_{by} - f_{bx}f_{ay}) + h_{aey}h_b{}^e{}_x - h_{bey}h_a{}^e{}_x, \tag{3.11}$$

where we have used (1.1) and (1.2).

We assume that the normal connection of N is flat, that is, $R_{abxy}=0$. Then we have

$$h_{aey}h_b{}^e{}_x - h_{bey}h_a{}^e{}_x = \frac{k}{4}(f_{ax}f_{by} - f_{bx}f_{ay}). \tag{3.12}$$

If the f-structure is parallel in the normal bundle, then we obtain from (3.4), (3.12) and (1.2) that

$$\begin{aligned} h_{ad}{}^x h_{bcx} - h_{bd}{}^x h_{acx} &= -\frac{k}{4}(g_{ad}g_{bc} - g_{ac}g_{bd} - \eta_b\eta_c g_{ad} + \eta_a\eta_c g_{bd} \\ &\quad + \eta_b\eta_d g_{ac} - \eta_a\eta_d g_{bc}), \end{aligned} \tag{3.13}$$

which and (3.7) imply $R_{abcd}=0$. Thus we have

THEOREM 9. *Let N be an n-dimensional tangential anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $M(k)$. If the f-structure in the normal bundle is parallel and the normal connection is flat, then N is locally flat.*

For $(m+1)$-dimensional submanifold we have the following

THEOREM 10. *Let N be an $(m+1)$-dimensional anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $M(k)$. If the connection in* *the normal bundle is flat, then N is locally flat.*

4. Tangential anti-invariant submanifolds of cosymplectic manifolds with vanishing cosymplectic Bochner curvature tensor

Suppose that M is a $(2m+1)$-dimensional cosymplectic manifold with vanishing cosymplectic Bochner curvature tesor. Then (1. 4) and lemma of chapter I imply

$$(4.1)\quad \begin{aligned} &K_{kjih}+(g_{kh}-\eta_k\eta_h)L_{ji}-(g_{jh}-\eta_j\eta_h)L_{ki}+L_{kh}(g_{ji}-\eta_j\eta_i)\\ &-L_{jh}(g_{ki}-\eta_k\eta_i)+\phi_{kh}M_{ji}-\phi_{jh}M_{ki}+M_{kh}\phi_{ji}-M_{jh}\phi_{ki}\\ &-2(M_{kj}\phi_{ih}+\phi_{kj}M_{ih})=0 \end{aligned}$$

and

$$(4.2)\quad \begin{aligned} &\nabla_k L_{ji}-\nabla_j L_{ki}+\frac{1}{8(m+1)(m+2)}(\phi_k{}^t\phi_{ji}-\phi_j{}^t\phi_{ki}\\ &-2\phi_i{}^t\phi_{kj})\nabla_t K=0. \end{aligned}$$

Let N be an $(n+1)$-dimensional tangential anti-invariant submanifold of M with vanishing cosymplectic Bochner curvature tensor. Transvecting (4. 1) with $B_a{}^kB_b{}^jB_c{}^jB_d{}^h$, we find

$$(4.3)\quad \begin{aligned} &R_{abcd}-h_{adx}h_{bc}{}^x+h_{bdx}h_{ac}{}^x+(g_{ad}-\eta_a\eta_d)L_{ji}B_b{}^jB_c{}^i\\ &-(g_{bd}-\eta_b\eta_d)L_{ki}B_a{}^kB_c{}^i+L_{kh}B_a{}^kB_d{}^h(g_{bc}-\eta_b\eta_c)\\ &-L_{jh}B_b{}^jB_d{}^h(g_{ac}-\eta_a\eta_c)=0, \end{aligned}$$

where we have used (2. 7) of chapter I, $\phi_{ji}B_b{}^jB_a{}^i=0$ and (1. 1).

We assume that the submanifold is η-umbilical, that is,

$$h_{bcx}=\alpha_x(g_{bc}-\eta_b\eta_c).$$

Then (4. 3) can be rewritten by

$$(4.4)\quad \begin{aligned} &R_{abcd}-\alpha_x\alpha^x(g_{ad}g_{bc}-g_{bd}g_{ac}-g_{ad}\eta_b\eta_c-\eta_a\eta_d g_{bc}+g_{bd}\eta_a\eta_c+\eta_b\eta_d g_{ac})\\ &+(g_{ad}-\eta_a\eta_d)L_{ji}B_b{}^jB_c{}^i-(g_{bd}-\eta_b\eta_d)L_{ki}B_a{}^kB_c{}^i\\ &+L_{kh}B_a{}^kB_d{}^h(g_{bc}-\eta_b\eta_c)-L_{jh}B_b{}^jB_d{}^h(g_{ac}-\eta_a\eta_c)=0. \end{aligned}$$

Since the unit vector field η^a is parallel with respect to the connection of N, N is locally a direct product of an N^n and N^1 generated by η^a and moreover N^n is a totally geodesic hypersurface in N([10]). We now suppose that the local expression of the hypersurface N^n in N is

$$(4.5)\quad y^a=y^a(z^s),$$

where z^s are local coordinates in N^n. The indices. s, t, u, v run over the range $\{1, 2, \ldots\ldots, n\}$. Differentiate (4. 5) and put

$$(4.6)\quad B_s{}^a=\partial_s y^a,\quad \partial_s=\partial/\partial z^s,$$

which is, for each fixed index s, a local vector field tangent to N^n. These local vector fields $B_s{}^a$ span the tangent plane of N^n at each point of N^n

and are orthogonal to η^a because of (1. 10).

Transvecting (4. 4) with $B_s{}^a B_t{}^b B_u{}^c B_v{}^d$, we find

$$R^*{}_{stuv}-\alpha_x\alpha^x(g_{sv}g_{tu}-g_{tv}g_{su})+g_{sv}L_{ji}B_t{}^j B_u{}^i -g_{tv}L_{ki}B_s{}^k B_u{}^i+L_{kh}B_s{}^k B_v{}^h g_{tu} -L_{jh}B_t{}^j B_v{}^h g_{su}=0, \tag{4. 7}$$

where $R^*{}_{stuv}$ is the curvature tensor of N^n, g_{sv} the induced metric on N^n from g_{ab} and $B_s{}^h=B_s{}^a B_a{}^h$.

If we put

$$C_{tu}=L_{ji}B_t{}^j B_u{}^i-\frac{1}{2}\alpha_x\alpha^x g_{tu}, \tag{4. 8}$$

then (4. 7) can be rewritten as

$$R^*{}_{stuv}+g_{sv}C_{tu}-g_{tv}C_{su}+C_{sv}g_{tu}-C_{tv}g_{su}=0. \tag{4. 9}$$

Transvecting (4. 9) with g^{sv}, we obtain

$$R^*{}_{tu}+(n-2)C_{tu}+C_s{}^s g_{tu}=0, \tag{4. 10}$$

where $R^*{}_{tu}$ is the Ricci tensor of N^n.

Transvecting (4. 10) with g^{tu}, we have

$$C_s{}^s=-\frac{1}{2(n-1)}R^*, \tag{4. 11}$$

where R^* denotes the scalar curvature of N^n. Therefore the equations (4. 10) and (4. 11) imply

$$C_{tu}=-\frac{1}{n-2}R^*{}_{tu}+\frac{1}{2(n-1)(n-2)}R^* g_{tu}, \tag{4. 12}$$

which and (4. 9) show that the Weyl conformal curvature tensor $C^*{}_{stuv}$ of N^n vanishes identically ($4\leqq n\leqq m$). Thus we have

THEOREM 11. *Let N be an $(n+1)$-dimensional ($n\geqq 4$) η-umbilical, tangential anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M with vanishing cosymplectic Bochner curvature tensor. Then N is locally a product of a conformally flat Riemannian manifold N^n and a 1-dimensional space N^1.*

We shall now seek out another condition in order to obtain the same conclusion as theorem 11. Let N be an $(n+1)$-dimensional ($n\geqq 4$) tangential anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M with vanishing cosymplectic Bochner curvature tensor. If the second fundamental tensors of N are commutative and the f-structure in the normal bundle is parallel, then from (3. 6) and (4. 3) we have

$$R_{abcd}+(g_{ad}-\eta_a\eta_d)L_{ji}B_b{}^j B_c{}^i-(g_{bd}-\eta_b\eta_d)L_{ki}B_a{}^k B_c{}^i +L_{kh}B_a{}^k B_d{}^h(g_{bc}-\eta_b\eta_c)-L_{jh}B_b{}^j B_d{}^h(g_{ac}-\eta_a\eta_c)=0. \tag{4. 13}$$

Transvecting (4.13) with $B_s^aB_t^bB_u^cB_v^d$, we find

$$R^*{}_{stuv}+g_{sv}D_{tu}-g_{tv}D_{su}+D_{sv}g_{tu}-D_{tv}g_{su}=0, \tag{4.14}$$

where

$$D_{tu}=L_{ji}B_t^jB_u^i.$$

Thus we have the following theorem by similar arguments of (4.10), (4.11) and (4.12).

THEOREM 12. *Let N be an $(n+1)$-dimensional $(n\geqq 4)$ anti-invariant tangential submanifold of a $(2m+1)$-dimensional cosymplectic manifold M with vanishing cosymplectic Bochner curvature tensor. If the f-structure in the norml bundle is parallel and the second fundamental tensors of N are commutative, then N is locally a product of a conformally flat Riemannian manifold N^n and 1-dimensional space N^1.*

Since $q_x{}^y=0$ for $n=m$, *i.e.*, dim $N=m+1$, we have (4.13). Hence we have the following

THEOREM 13. *Let N be an $(m+1)$-dimensional $(m\geqq 4)$ anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M with vanishing cosymplectic Bochner curvature tensor. If the second fundamental tensors of N are commutative, then N is locally a product of a conformally flat Rimannian manifold N^m and 1-diemensional space N^1*

In the sequel, we study the case that dim $N=4$, i.e. $n=3$.

THEOREM 14. *Let N be a 4-dimensional tangential anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold with vanishing cosymplectic Bochner curvature tensor. If N is η-umbilical, then N is locally a product of a conformally flat Riemannian manifold N^3 and a 1-dimensional space N^1.*

Proof. Since M has vanshing cosymplectic Bochner curvature tensor, we have, from (4.2),

$$(\nabla_k L_{ji}-\nabla_j L_{ki})B_a^kB_b^jB_c^j=0. \tag{4.15}$$

If we put $L_{bc}=L_{ji}B_b^jB_c^i$, then we obtain

$$\nabla_a L_{bc}=B_a^kB_b^jB_c^i\nabla_k L_{ji}+L_{ji}\alpha^x(g_{ab}-\eta_a\eta_b)C_x^jB_c^i$$
$$+L_{ji}B_b^j(g_{ac}-\eta_a\eta_c)\alpha^xC_x^i,$$

from which and (4.15) we find

$$\nabla_a L_{bc}-\nabla_b L_{ac}+L_a(g_{bc}-\eta_b\eta_c)-L_b(g_{ac}-\eta_a\eta_c)=0, \tag{4.16}$$

where $L_b=L_{ji}B_b^jC_x^i\alpha^x$.

If we put $L_{tu}=L_{bc}B_t^bB_u^c$, then we have

$$\nabla_s L_{tu}=B_s^aB_t^bB_u^c\nabla_a L_{bc}.$$

The last equation and (4.16) imply

(4.17) $$\nabla_s L_{tu} - \nabla_t L_{su} + L_s g_{tu} - L_t g_{su} = 0,$$

where $L_t = L_b B_t{}^b$.

Since $L_{tu} = C_{tu} + \frac{1}{2}\alpha_x \alpha^x g_{tu}$, we obtain

(4.18) $$\nabla_s L_{tu} = \nabla_s C_{tu} + g_{tu}(\nabla_s \alpha_x)\alpha^x.$$

Substituting (4.18) into (4.17), we find

(4.19) $$\nabla_s C_{tu} - \nabla_t C_{su} + (\alpha^x \nabla_s \alpha_x + L_s) g_{tu} - (\alpha^x \nabla_t \alpha_x + L_t) g_{su} = 0.$$

Transvecting (4.19) with g^{tu}, we have

(4.20) $$\alpha^x \nabla_s \alpha_x + L_s = 0$$

by the help of (4.11), (4.12) and the second Bianchi identity. Hence we have

(4.21) $$\nabla_s C_{tu} - \nabla_t C_{su} = 0,$$

from which and (4.12) we find $C^*{}_{stu} = 0$. This completes the proof.

III. Normal anti-invariant submanifolds of cosymplectic manifolds

In this chapter we investigate anti-invariant submanifolds, which are normal to the structure vector field, of cosymplectic manifolds. We call such a submanifold normal anti-invariant submanifold.

1. Basic formulas of normal anti-invariant submanifolds

Let N be an n-dimensional normal anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M $(n \leqq m)$. Then we have

$$f_c{}^a = 0 \ \textit{and}\ \eta^a = 0.$$

Therefore the equations of chapter I which contain $f_c{}^a$ and η^a can be rewritten as

(1.1) $\eta^h = \eta^x C_x{}^h$,	(1.2) $f_c{}^x f_x{}^b = \delta_c{}^b$,
(1.3) $f_c{}^x q_x{}^y = 0$,	(1.4) $q_x{}^y q_y{}^z = -\delta_x{}^z + \eta_x \eta^z + f_x{}^a f_a{}^z$,
(1.5) $f_x{}^b \eta^x = 0$, $q_x{}^y \eta_y = 0$,	(1.6) $\eta_x \eta^x = 1$,
(1.7) $f_a{}^x f_x{}^a = n$,	(1.8) $q_{xy} q^{xy} = 2(m-n)$,
(1.9) $h_b{}^a{}_x \eta^x = 0$,	(1.10) $\nabla_b \eta_x = 0$,
(1.11) $h_{cb}{}^x f_x{}^a = f_c{}^x h_b{}^a{}_x$	(1.12) $\nabla_b f_c{}^y = -h_{bc}{}^x q_x{}^y$.

If we transvect the equation (1.4) with $q_w{}^x$ and take account of (1.3) and (1.5), then we have

$$q_w{}^x q_x{}^y q_y{}^z + q_w{}^z = 0.$$

If the dimension of N is less than m, then, from (1.8), $q_x{}^y$ does not vanish and hence $q_x{}^y$ defines an f-structure in the normal bundle of N. If

N is an m-dimensional normal anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M, then $q_x{}^y$ vanishes identically.

2. Normal anti-invariant submanifolds of $M(k)$

Let N be an n-dimensional normal anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $M(k)$ of constant ϕ-holomorphic sectional curvature k. Then the equation (2. 7) of Gauss and (1. 7) of chapter I imply

$$(2.1)\qquad R_{abcd}=-\frac{k}{4}(g_{ac}g_{bd}-g_{bc}g_{ad})+h_{ad}{}^x h_{bcx}-h_{bd}{}^x h_{acx},$$

where we have used $\phi_{ij}B_a{}^jB_b{}^j=0$ and $\eta^hB_{ah}=0$.

If N is totally geodesic, then we have

$$R_{abcd}=-\frac{k}{4}(g_{ac}g_{bd}-g_{bc}g_{ad}),$$

which shows that N is of constant curvature $\frac{k}{4}$

Transvecting (2. 1) with g^{ab}, we obtain

$$(2.2)\qquad R_{bc}=\frac{k}{4}(n-1)g_{bc}+h_a{}^{ax}h_{bcx}-h_b{}^e{}_x h_{ec}{}^x,$$

from which we find

$$(2.3)\qquad R=\frac{k}{4}n(n-1)+h_a{}^a{}_x h_b{}^{bx}-h_{bex}h^{bex}.$$

Moreover, if the submanifold N is minimal, then (2. 2) and (2. 3) imply

$$(2.4)\qquad R_{bc}=\frac{k}{4}(n-1)g_{bc}-h_b{}^e{}_x h_{ec}{}^x$$

and

$$(2.5)\qquad R=\frac{k}{4}n(n-1)-h_{bex}h^{bex}.$$

From the equations (2. 1), (2. 4) and (2. 5) we obtain the following proposition.

PROPOSITION 15. *Let N be an n-dimensional normal anti-invariant minimal submanifold of a cosymplectic manifold $M(k)$ of constant ϕ-holomorphic sectional curvature k. Then N is totally geodesic if and only if N satisfies one of the following conditions*

(*a*) *N is of constant curvature* $\frac{k}{4}$,

(*b*) $R_{bc}=\frac{k}{4}(n-1)g_{bc}$,

(*c*) $R=\frac{k}{4}n(n-1)$.

REMARK. Proposition 15 corresponds to the case of anti-invariant submanifold of a complex space form ([3]).

3. Normal anti-invariant submanifolds with parallel f-structure in the normal bundle

In section 1 we have shown that $q_x{}^y$ defines an f-structure in the normal bundle. In this section we investigate normal anti-invariant submanifolds with parallel f-structure in the normal bundle.

Let N be an n-dimensional normal anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M. Suppose that the f-structure $q_x{}^y$ is parallel in the normal bundle of N, that is, $\nabla_b q_x{}^y=0$. Then the equation (2. 29) of chaptet I implies

(3. 1) $$h_b{}^e{}_x f_e{}^z - f_x{}^e h_{eb}{}^z = 0.$$

Transvecting (3. 1) with $f_c{}^x$ and taking account of (1. 2), we obtain

(3. 2) $$h_{bc}{}^y = h_b{}^e{}_x f_c{}^x f_e{}^y.$$

From (3. 2) and (1. 2) we have

(3. 3) $$h_{ad}{}^x h_{bcx} = h_a{}^e{}_z f_d{}^z f_e{}^x h_b{}^g{}_y f_c{}^y f_{gx} = h_a{}^e{}_z h_{eby} f_d{}^z f_c{}^y.$$

From (3. 3) we obtain the following lemma.

LEMMA 16. *Let N be a normal anti-invariant submanifold of a cosymplectic manifold. If the f-structure $q_x{}^y$ is parallel in the normal bundle, then we have*

(3. 4) $$h_{ad}{}^x h_{bcx} - h_{bd}{}^x h_{acx} = (h_a{}^e{}_z h_{ebw} - h_b{}^e{}_z h_{eaw}) f_d{}^z f_c{}^w.$$

We shall now prove the following

THEOREM 17. *Let N be an n-dimensional normal anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $M(k)$ and let the f-structure $q_x{}^y$ is parallel in the normal bundle of N. Then N is of constant curvature $\frac{k}{4}$ if and only if the second fundamental tensors of N are commutative.*

Proof. Suppose that N is of constant curvature $\frac{k}{4}$. Then the equation (2. 1) implies

$$h_{ad}{}^x h_{bcx} - h_{bd}{}^x h_{acx} = 0,$$

from which and (3. 4) we obtain

(3. 5) $$(h_a{}^e{}_z h_{ebw} - h_b{}^e{}_z h_{eaw}) f_d{}^z f_c{}^w = 0.$$

Transvecting (3. 5) with $f^d{}_x$, we have

(3. 6) $$(h_a{}^e{}_z h_{ebw} - h_b{}^e{}_z h_{eaw})(q_x{}^y q_y{}^z + \delta_x{}^z - \eta_x \eta^z) f_c{}^w = 0$$

by the help of (1. 4).

On the other hand, we have, by the help of (3. 2),

$$h_{bc}{}^{y}q_{yz}=h_{b}{}^{e}{}_{x}f_{c}{}^{x}f_{e}{}^{y}q_{yz},$$

which and (1. 3) imply

$$h_{bc}{}^{y}q_{yz}=0. \tag{3.7}$$

From (3. 7) and (1. 9), (3. 6) can be reduced to

$$(h_{a}{}^{e}{}_{x}h_{ebw}-h_{b}{}^{e}{}_{x}h_{eaw})f_{c}{}^{w}=0. \tag{3.8}$$

Transvecting (3. 8) with $f^{c}{}_{y}$ and by the similar arguments, we obtain

$$h_{a}{}^{e}{}_{x}h_{eby}-h_{b}{}^{e}{}_{x}h_{eay}=0,$$

which shows that the second fundamental tensors of N are commutative.

Conversely, suppose that the second fundamental tensors of N are commutative. Then from lemma 16 we have

$$h_{ad}{}^{x}h_{bcx}-h_{bd}{}^{x}h_{acx}=0.$$

Substituting the last equation into (2. 1), we have

$$R_{abcd}=-\frac{k}{4}(g_{ac}g_{bd}-g_{bc}g_{ad}),$$

which shows that the submanifold N is of constant curvature $\frac{k}{4}$. This completes the proof of the theorem.

Let us consider an m-dimensional normal anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M. Then $q_{x}{}^{y}=0$ because of (1. 8). In this case we also have the equations (3. 1) and (3. 4). Hence the theorem 17 is satisfied without the assumption with respect to f-structure. Thus we have

THEOREM 18. *Let N be an m-dimensional normal anti-inveriant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $M(k)$. Then N is of constant curvature $\frac{k}{4}$ if and only if second fundamental tensors of N are commutative.*

We now consider the normal connection of an n-dimensional normal anti-invariant submanifold of a cosymplectic manifold $M(k)$ of dimension $(2m+1)$. Substituting (1. 7) of chapter I into the equation of Ricci (chapter I. (2. 9)), we obtain

$$R_{bcxy}=h_{bey}h_{c}{}^{e}{}_{x}-h_{cey}h_{b}{}^{e}{}_{x}-\frac{k}{4}(f_{bx}f_{cy}-f_{cx}f_{by}), \tag{3.9}$$

where we have used (1. 1) and (1. 2).

We assume that the normal connection of N is flat, that is, $R_{abxy}=0$ Then we obtain

(3. 10) $$h_{bey}h_c{}^e{}_x - h_{cey}h_b{}^e{}_x = \frac{k}{4}(f_{bx}f_{cy} - f_{cx}f_{by}).$$

If the f-structure is parallel in the normal bundle, then (3. 4) and (3. 10) imply

$$h_{ad}{}^x h_{bcx} - h_{bd}{}^x h_{acx} = \frac{k}{4}(g_{ac}g_{bd} - g_{bc}g_{ad}),$$

from which and (2. 1) we get $R_{abcd}=0$. Thus we have

THEOREM 19. *Let N be an n-dimensional normal anti-invariant submanifold of $M(k)$ with parallel f-structure in the normal bundle. If the connection in the normal bundle is flat, then N is locally flat.*

4. A normal anti-invariant submanifold of a cosymplectic manifold with vanishing cosymplectic Bochner curvature tensor

As a theorem giving a relation between the Weyl conformal curvature tensor and the contact Bochner curvature tensor, K. Yano ([15]) proved the following

THEOREM. *Let N be an n-dimensional ($n \geqq 3$) totally umbilical, anti-invariant submanifold normal to the structure vector field of a $(2m+1)$-dimensional Sasakian manifold M with vanishing contact Bochner curvature tensor. Then N is conformally flat.*

In this section we investigate a normal anti-invariant submanifold of a cosymplectic manifold with vanishing cosymplectic Bochner curvature tensor and obtain a theorem corresponding to the above theorem.

Let N be an n-dimensional normal anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M with vanishing cosymplectic Bochner curvature tensor. Then from (1. 4) and lemma of chapter I we have respectively

(4. 1) $$\begin{aligned} K_{kjih} &+ (g_{kh}-\eta_k\eta_h)L_{ji} - (g_{jh}-\eta_j\eta_h)L_{ki} \\ &+ L_{kh}(g_{ji}-\eta_j\eta_i) - L_{jh}(g_{ki}-\eta_k\eta_i) + \phi_{kh}M_{ji} \\ &- \phi_{jh}M_{ki} + M_{kh}\phi_{ji} - M_{jh}\phi^{ki} - 2(M_{kj}\phi_{ih} + \phi_{kj}M_{ih}) = 0, \end{aligned}$$

(4. 2) $$\nabla_k L_{ji} - \nabla_j L_{ki} + \frac{1}{8(m+1)(m+2)}(\phi_k{}^t\phi_{ji} - \phi_j{}^t\phi_{ki} - 2\phi_{kj}\phi_i{}^t)\nabla_t K = 0.$$

We now assume that the submanifold N is totally umbilical, that is, $h_{bc}{}^x = \alpha^x g_{bc}$. Transvecting (4. 1) with $B_a{}^k B_b{}^j B_c{}^i B_d{}^h$, we find

(4. 3) $$\begin{aligned} R_{abcd} &- (g_{ad}g_{bc} - g_{bd}g_{ac})\alpha_x\alpha^x + g_{ad}L_{ji}B_b{}^j B_c{}^i \\ &- g_{bd}L_{ki}B_a{}^k B_c{}^i + L_{kh}B_a{}^k B_d{}^h g_{bc} - L_{jh}B_b{}^j B_d{}^h g_{ac} = 0 \end{aligned}$$

by virtue of (2. 7) of chapter I, (1. 1) and $\phi_{ji}B_c^jB_b^i=0$.

If we put

$$C_{bc}=L_{ji}B_b^jB_c^i-\frac{1}{2}\alpha_x\alpha^x g_{bc},$$

then (4. 3) can be rewritten as

(4. 4) $$R_{abcd}+g_{ad}C_{bc}-g_{bd}C_{ac}+C_{ad}g_{bc}-C_{bd}g_{ac}=0.$$

Transvecting (4. 4) with g^{ab}, we obtain

(4. 5) $$C_{bc}=-\frac{1}{n-2}R_{bc}+\frac{1}{2(n-1)(n-2)}Rg_{bc}.$$

Therefore (4. 4) and (4. 5) show that the Weyl conformal curvature tensor of N vanishes identically.

Transvecting (4. 2) with $B_a^kB_b^jB_c^i$, we obtain

(4. 6) $$(\nabla_k L_{ji}-\nabla_j L_{ki})B_a^kB_b^jB_c^i=0$$

by virtue of $\phi_{ji}B_b^jB_c^i=0$.

If we put $L_{bc}=L_{ji}B_b^jB_c^i$ and $L_c=L_{ji}B_c^jC_x^j\alpha^x$,

then we have

$$\nabla_a L_{bc}=(\nabla_k L_{ji})B_a^kB_b^jB_c^i+L_c g_{ab}+L_b g_{ac},$$

from which and (4. 6) we have

(4. 7) $$\nabla_a L_{bc}-\nabla_b L_{ac}+L_a g_{bc}-L_b g_{ac}=0.$$

Since $L_{bc}=C_{bc}+\frac{1}{2}\alpha_x\alpha^x g_{bc}$ we find

(4. 8) $$\nabla_a L_{bc}=\nabla_a C_{bc}+g_{bc}(\nabla_a\alpha_x)\alpha^x.$$

Substituting (4. 8) into (4. 7), we find

(4. 9) $$\nabla_a C_{bc}-\nabla_b C_{ac}+(\alpha^x\nabla_a\alpha_x+L_a)g_{bc}-(\alpha^x\nabla_b\alpha_x+L_b)g_{ac}=0.$$

Transvecting (4. 9) with g^{bc}, we have

$$L_a+\alpha^x\nabla_a\alpha_x=0$$

by the help of (4. 5) and $\nabla_b R_a^b=\frac{1}{2}\nabla_a R$ Therefore(4. 9) can be rewritten as

$$\nabla_a R_{bc}-\nabla_b R_{ac}-\frac{1}{2(n-1)}(g_{bc}\nabla_a R-g_{ac}\nabla_b R)=0.$$

from which we find $C_{abc}=0$. Thus we have the folowing

THEOREM 20. *Let N be an n-dimensional normal totally umbilical anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic manifold M with vanishing cosymplectic Bochner curvature tensor $(3\leqq n\leqq m)$. Then N is conformally flat.*

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