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ON DIRECT LIMITS AND PRODUCTS OF RINGS OF QUOTIENTS

MUN-GU SOHN AND SUK GEUN HWANG*

§1. Introduction

Let R be an associative ring with a nonzero identity element 1, and let ${}_R\mathscr{M}$ denote the category of all left R-modules and R-homomorphisms. For a given torsion radical σ on ${}_R\mathscr{M}$ the ring of quotients $Q_\sigma(R)$ of R *w.r.t.* σ is defined by a direct limit [1]

$$Q_\sigma(R)=\varinjlim \mathrm{Hom}_R(U, R/\sigma(R)),\quad U\in\mathscr{F}_\sigma$$

where $\mathscr{F}_\sigma$ is the associated filter of σ and the direct limit is taken over the downwards directed family $\mathscr{F}_\sigma$ of left ideals([1], [5]).

A more general form of a ring of quotients can be found in [2]. A ring Q is called a ring of quotients of R *w.r.t.* σ if R is a subring of Q such that ${}_RR$ is a large σ-submodule of ${}_RQ$ [2].

In this paper, we determine the structure of the direct limit $\varinjlim Q_{\sigma_i}(R)$ of the rings of quotients of a left noetherian ring R with $\{\sigma_i \mid i\in I\}$ being an increasing family of torsion radicals on ${}_R\mathscr{M}$ and I being a right directed preordered set. And we prove that the ring of quotients of a product ring is isomorphic to the product of rings of quotients of the factor rings by constructing some torsion radical for the product ring.

§2. Some preliminaries

As is in the previous section, let R be an associative ring with a nonzero identity element 1 and let ${}_R\mathscr{M}$ denote the category of all left R-modules and R-homomorphisms. By a *torsion radical* on ${}_R\mathscr{M}$ [4, p. 5] we mean an object function $\sigma: {}_R\mathscr{M}\to{}_R\mathscr{M}$ which satisfies the following conditions:

(i) $\sigma(M)$ is a submodule of M for every R-module M;

(ii) whenever $f: M\to N$ is an R-homomorphism, $f(\sigma(M))\subset\sigma(N)$;

(iii) $\sigma(M/\sigma(M))=0$, for every R-module M;

(iv) for all R-modules M and N such that N is a submodule M, we have $\sigma(N)=N\cap\sigma(M)$.

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Also, by a *Gabriel filter* (or merely a *filter*) we mean a nonempty family $\mathscr{F}$ of left ideals of R which has the properties:

(i) if $U\in\mathscr{F}$, and if V is a left ideal of R such that $V\supset U$, then $V\in\mathscr{F}$;

(ii) if $U, V\in\mathscr{F}$, then $U\cap V\in\mathscr{F}$;

(iii) if $U\in\mathscr{F}, x\in R$, then there is a $V\in\mathscr{F}$ such that $Vx\subset U$.

It is known that there is an one to one correspondence between torsion radicals and the filters. Let σ be a torsion radical on ${}_R\mathscr{M}$. and T_σ be its associated filter. A left R-module M is called to be *σ-torison* (*σ-torsion free*) if $\sigma(M)=M$ (resp. $\sigma(M)=0$). A submodule N of a module M is called a *σ-submodule* if M/N is σ-torsion. A left R-module E is called to be *σ-injec tive* if, whenever N is a σ-submodule of a left R-module M, every homom orphism $f: N\to E$ can be extended to a homomorphism $g: M\to E$. Moreover, if the existence of such a g is unique, E is called to be *faithfully σ-injective.* For a torsion radical σ, the homomorphism $R\to Q_\sigma(R)$ will be denoted by ι_σ which comes to be a monomorphism if R is σ-torsion free.

For two torsion radicals σ and τ, there is defined $\sigma\leq\tau$ if and only if $\sigma(M)\subset\tau(M)$ for every module M, and under this order, any set of torsion radicals on ${}_R\mathscr{M}$ has an infimum in the set of all torsion radicals [1]. With this definition in mind, we know immediately that if M is σ-torsion (τ-torison free), then it is τ-torsion (σ-torsion free) whenever $\sigma\leq\tau$.

§3. The direct limits of rings of quotients

Throughout this section, let I be a right directed preordered set, and $\{\sigma_j | i\in I\}$ be an increasing family of torsion radicals on ${}_R\mathscr{M}$. The torsion radical $\sup_{i\in I}\sigma_i$ will be denoted by σ_o and the associated filters $\mathscr{F}_{\sigma_i}$ and $\mathscr{F}_{\sigma_o}$ of σ_i and σ_o will be denoted by $\mathscr{F}_j$ and $\mathscr{F}_o$ for the brevity.

In the following, we prepare some lemmas which tells the relationship between the direct limit of a direct system of modules and each factor of the system.

LEMMA 1. *Let (M_i, ϕ_{ji}) be a direct system of R-modules over I. Then $\varinjlim M_i$ is σ_o-torsion (σ_o-torsion free) if, for each $i\in I$, M_i is σ_i-torsion (σ_i-torsion free resp.).*

Proof. Suppose that M_i is σ_i-torsion for each $i\in I$, and let $x\in\varinjlim M_i$ be such that $x\neq 0$. Then, for the restriction ϕ_i of $\bigoplus_{i\in I} M_i\to\varinjlim M_i$ to M_i, there is some $i\in I$ and some $x_i\in M_i$ such that $x=\phi_i(x_i)$. Since $\sigma_i(M_i)=M_i$, there is an ideal $U\in\mathscr{F}_i$ such that $Ux_i=0$, so that $Ux=U\phi_i(x_i)=\phi_i(Ux_i)=0$. But, since $U\in\mathscr{F}_o$ also, we have that $\varinjlim M_i$ is σ_o-torsion.

Next suppose that $\sigma_i(M_i)=0$ for every $i\in I$ and that $x=\phi_i(x_i)\neq 0$ be an arbitrary element of $\varinjlim M_i$ with $x_i\in M_i$ for some $i\in I$. Then $\phi_j(x_i)\neq 0$ for all $j\geq i$. Let $U\in\mathcal{F}_0$ then $U\in\mathcal{F}_k$ for some $k\in I$. Let $r\in I$ be such that i, $k\leq r$, then $x=\phi_r(x_r)$ where $x_r=\phi_{ri}(x_i)\in M_r$. Since, for all $s\in I$ such that $s\geq r$, $U\in\mathcal{F}_s$ and $\sigma_s(M_s)=0$, we have, for all $s\geq r$, that $\phi_{sr}(Ux_r)=U\phi_{sr}(x_r)\neq 0$ so that $Ux=U\phi_r(x_r)=\phi_r(Ux_r)\neq 0$.

Moreover, if R is left noetherian, then we know that the σ_0-injectivity of the direct limit follows from the σ_i-injectivity of each M_i.

LEMMA 2. *Let R be left noetherian, and let (M_i, ϕ_{ji}) be a direct system of R-modules over I. Then $\varinjlim M_i$ is σ_0-injective if, for each $i\in I$, M_i is σ_i-injective.*

Proof. Let $U\in\mathcal{F}_0$ and $f\in\mathrm{Hom}_R(U, \varinjlim M_i)$. It suffices to show that f can be extended to a homomorphism $R\rightarrow\varinjlim M_i$ by [1]. For each $i\in I$, let ϕ_i be the restriction of $\bigoplus_{i\in I} M_i\rightarrow\varinjlim M_i$ to M_i. Since U is finitely generated, $f(U)\subset\phi_j(M_j)$ for some $j\in I$. Let M_j' be a finitely generated submodule of M_j such that $\phi_j(M_j')=f(U)$, and let ϕ_j' be the restriction of ϕ_j to M_j', then $K_j=\phi_j'^{-1}(0)$ is also finitely generated. Since $\phi_j(K_j)=0$, there is some $i_0\in I$ such that $\phi_{ij}(K_j)=0$ for all $i\geq i_0$. Since $U\in\mathcal{F}_0$, there is some $k\in I$ such that $U\in\mathcal{F}_k$. Let $r\in I$ be such that i_0, $k\leq r$, then $U\in\mathcal{F}_r$. Since M_r is σ_r-injective, the homomorphism f which may be considered as a homomorphism $U\rightarrow M_r$ by th fact that $\phi_{rj}(K_j)=0$ can be extended to a homomorphism $g: R\rightarrow M_r$. And we may consider g as a homomorphism $R\rightarrow\varinjlim M_i$.

From the above lemmas, we know easily that if σ is a (fixed) torsion radical, then the direct limit of σ-torsion (σ-torison free) modules is σ-torsion (σ-torsion free resp.), and that if R is left noetherian, then the direct limit of σ-injective modules is σ-injective.

Now, we are to construct the direct limit of rings of quotients with the aid of the following results. Before entering our discussion, recall that the homomorphism $R\rightarrow Q_\sigma(R)$ is denoted by ι_σ.

LEMMA 3. *Let σ and τ be torsion radicals on ${}_R\mathcal{M}$ such that $\sigma\leq\tau$. Then there is a unique homomorphism $\phi_{\tau\sigma}: Q_\sigma(R)\rightarrow Q_\tau(R)$ such that $\phi_{\tau\sigma}\iota_\sigma=\iota_\tau$.*

Proof. This result follows from the fact that $Q_\tau(R)$ is faithfully τ-injective and that the σ-submodule $R/\sigma(R)$ of $Q_\sigma(R)$ is also a τ-submodule of $Q_\sigma(R)$.

By the uniqueness of such a homomorphism $\phi_{\tau\sigma}: Q_\sigma(R)\to Q_\tau(R)$, we have also easily the following

COROLLARY. *Let σ, τ, μ be torsion radicals on ${}_R\mathscr{M}$ such that $\sigma\le\tau\le\mu$. Then we have $\phi_{\mu\sigma}=\phi_{\mu\tau}\phi_{\tau\sigma}$, where each of $\phi_{\mu\sigma}, \phi_{\mu\tau}$ and $\phi_{\tau\sigma}$ is the unique homomorphism such as stated in the above lemma.*

For $i, j\in I$ with $i\le j$, $\sigma_i\le\sigma_j$ by the assumption on the set $\{\sigma_i \mid i\in I\}$. From now on, the homomorphism $\phi_{\sigma_j\sigma_i}: Q_{\sigma_i}(R)\to Q_{\sigma_j}(R)$ appeared in the Lemma 3 will be denoted by ϕ_{ji} for conveniences. Then, by the corollary to Lemma 3, we know that $(Q_{\sigma_i}(R), \phi_{ji})$ forms a direct system over I. Let's denote $Q_{\sigma_i}(R)\to\varinjlim Q_{\sigma_i}(R)$ by ϕ_i as usual.

Since, for a torsion radical σ, a left R-module is faithfully σ-injective if and only if it is σ-injective and σ-torsion free, we know that, for a left noetherian ring R, $\varinjlim Q_{\sigma i}(R)$ is faithfully σ_o-injective by the Lemmas 1 and 2.

THEOREM 1. *If R is a σ_o-torsion free left noetherian ring, then $\varinjlim Q_{\sigma_i}(R)$ is isomorphic to $Q_{\sigma_o}(R)$ in a unique way.*

Proof. Let, for each $i\in I$, the homomorphism $Q_{\sigma_i}(R)\to Q_{\sigma_o}(R)$ be denoted by ψ_i, and let $\psi_{ji}=\phi_{ji}$. Then, by the corollary to Lemma 3, $\{\psi_i \mid i\in I\}$ forms a direct system of homomorphisms so that $\phi=\varinjlim\psi_i$ is the unique homomorphism $\varinjlim Q_{\sigma_i}(R)\to Q_{\sigma_o}(R)$ which makes the following diagram commute for each $i\in I$.

$$\begin{array}{ccc} Q_{\sigma_i}(R) & \xrightarrow{\ \phi_i\ } & \varinjlim Q_{\sigma_i}(R) \\ & \psi_i\searrow \quad \swarrow\phi & \\ & Q_{\sigma_o}(R) & \end{array}$$

Since, for every $i, j\in I$, $\psi_i\iota_{\sigma_i}=\psi_j\iota_{\sigma_j}$, we can define a homomorphism $\eta: R\to\varinjlim Q_{\sigma_i}(R)$ by $\eta=\phi_i\iota_{\sigma_i}$ for each $i\in I$. Then, since $\varinjlim Q_{\sigma_i}(R)$ is faithfully σ_o-injective and R is a σ_o-submodule of $Q_{\sigma_o}(R)$, we can extend η to a unique homomorphism $\rho: Q_{\sigma_o}(R)\to\varinjlim Q_{\sigma_i}(R)$. But then $\phi\rho\iota_{\sigma_o}=\phi\eta=\phi\phi_i\iota_{\sigma_i}=\psi_i\iota_{\sigma_i}=\iota_{\sigma_o}$ for all $i\in I$ so that $\phi\rho$ is the identity homomorphism of $Q_{\sigma_o}(R)$ onto itself. By a similar computation, we get that $\rho\phi$ is the identity homomorphism of $\varinjlim Q_{\sigma_i}(R)$ onto itself. Therefore we know that ϕ is the uniquely determined isomorphism of $Q_{\sigma_o}(R)$ onto $\varinjlim Q_{\sigma_i}(R)$.

§ 4. Ring of quotients of a product ring

In this section, by a ring of quotients of R *w. r. t.* σ we shall mean a ring Q containing R as a subring such that ${}_RR$ is a large σ-submodule of ${}_RQ$ as left R-modules [2].

Some useful properties of rings of quotients can be found in [2]. We will state some of them as our lemmas.

LEMMA 4. ([2]) *Let* $Q_\sigma(R)=\varinjlim \mathrm{Hom}_R(U, R/\sigma(R))$, $U\in\mathscr{F}_\sigma$ *be the ring of quotients of* R *w. r. t.* σ *constructed by Goldman, O.* [1]. *Then any ring of quotients of* R *w. r. t.* σ *is a subring of* $Q_\sigma(R)$.

From such a point of view, we know that $Q_\sigma(R)$ is the maximal one in the class of all rings of quotients of R *w. r. t.* σ.

LEMMA 5. ([2]) *Let* Q *be any ring containing* R *as a subring. Then* Q *is isomorphic to* $Q_\sigma(R)$ *if and only if* Q *is a ring of quotients of* R *w. r. t.* σ *such that, for every* $U\in\mathscr{F}_\sigma$ *and every* $f\in\mathrm{Hom}_R(U, R)$, *there is a unique* $q\in Q$ *such that* $f(u)=uq$ *for all* $u\in U$.

Let $\{R_i\mid i\in I\}$ be a nonempty family of rings each of whose members has a nonzero identity element. For each $i\in I$, let σ_i denote a torsion radical on ${}_{R_i}\mathscr{M}$ such that $\sigma_i(R_i)=0$ and $\mathscr{F}_i$ be its associated filter. In the rest of this paper, we construct a torsion radical σ on ${}_R\mathscr{M}$ with $R=\prod_{i\in I} R_i$ from the family $\{\sigma_i\mid i\in I\}$, and will prove that $Q_\sigma(R)$ is, in fact, isomorphic to the product ring $\prod_{i\in I} Q_{\sigma_i}(R_i)$ of rings of quotients $Q_{\sigma_i}(R_i)$, $i\in I$.

Let's denote by κ_i and π_i the canonical injection $R_i\to R$ and the canonical projection $R\to R_i$ respectively. It can be easily checked that U is a left ideal of R if and only if, for each $i\in I$, $\pi_i(U)$ is a left ideal of R_i.

THEOREM 2. *The class* $\mathscr{F}=\{U=\prod_{i\in I}U_i\mid U_i\in\mathscr{F}_{\sigma_i}\}$ *forms a Gabriel filter on* R.

Proof. Let B be a left ideal of R containing some $U=\prod_{i\in I}U_i\in\mathscr{F}$. Then. for each $i\in I$, $U_i\subset\pi_i(B)=B_i$ so that $B_i\in\mathscr{F}_{\sigma_i}$. Thus we have that $B=\prod_{i\in I}B_i$ is a member of $\mathscr{F}$.

Next, since, clearly, $(\prod_{i\in I}U_i)\cap(\prod_{i\in I}B_i)=\prod_{i\in I}(U_i\cap B_i)$ for any $\prod_{i\in I}U_i$, $\prod_{i\in I}B_i\in\mathscr{F}$ and $\mathscr{F}_{\sigma_i}$ is a filter on R_i for each $i\in I$. we know that $\mathscr{F}$ is closed under the finite intersection. And, now, suppose that $r\in R$ and $U=\prod_{i\in I}U_i\in\mathscr{F}$. Then, for each $i\in I$, $U_i\in\mathscr{F}_{\sigma_i}$ and $r(i)\in R_i$, and hence there exists a $B_i\in\mathscr{F}_{\sigma_i}$ such that $B_ir(i)\subset U_i$ so that $(\prod_{i\in I}B_i)r\subset U$. But $B=\prod_{i\in I}B_i\in\mathscr{F}$. Therefore $\mathscr{F}$ is a filter on R.

Let's denote by σ the torsion radical on ${}_R\mathscr{M}$ corresponding to the filter $\mathscr{F}$ just constructed in above theorem.

THEOREM 3. *Let Q_i be a ring containing R_i as a subring, for each $i\in I$, so that $R=\prod_{i\in I} R_i$ is a subring of $Q=\prod_{i\in I} Q_i$. Then Q is a ring of quotients of R w.r.t. σ if and only if:*

(i) *Q_i/R_i is σ_i-torsion free for each $i\in I$;*

and

(ii) *for every nonzero $q\in Q$, there exists some $j\in I$ such that $R_jq(j)\cap R_j \neq 0$.*

Proof. (*Necessity*) Let $q_i\in Q_i$, then $q=\kappa_i(q_i)\in Q$. Thus there is a $U=\prod_{i\in I}U_i\in\mathscr{F}$ such that $Uq\subset R$, so that $U_iq_i\subset R_i$ showing us that (i) holds.

Next, let $0\neq q\in Q$. Then $Rq\cap R\neq 0$ since ${}_RQ$ is an essential extension of ${}_RR$. But this implies that there is some $j\in I$ such that $R_iq_j\cap R_j\neq 0$. Therefore we get (ii).

(*Sufficiency*) Assume (i) and (ii), and let $q\in Q$. Then $q(i)\in Q_i$ for each $i\in I$. Thus, by (i), there is some $U_j\in\mathscr{F}_{\sigma j}$ such that $U_iq(i)\subset R_i$. Now, the ideal $U=\prod_{i\in I}U_i\in\mathscr{F}$ satisfies the relation $Uq\subset R$. And for $0\neq q\in Q$, we have, by (ii), that $R_jq(j)\cap R_j\neq 0$ for some $j\in I$, which implies that $Rq\cap R\neq 0$ showing us that ${}_RQ$ is an essential extension of ${}_RR$.

Since any ring of quotients of a ring *w.r.t.* a torsion radical contains the original ring as a subring, we have easily the following

COROLLARY. *Let for each $i\in I$, Q_i be a ring of quotients of R_i w.r.t. σ_i. Then $Q=\prod_{i\in I}Q_i$ is a ring of quotients of $R=\prod_{i\in I} R_i$ w.r.t. σ.*

With the aid of the above Corollary and the Lemma 5, we are to give one of our main theorems in the following

THEOREM 4. *$Q_\sigma(\prod_{i\in I} R_i)$ is isomorphic to $\prod_{i\in I} Q_{\sigma_i}(R_i)$.*

Proof. Let $R=\prod_{i\in I} R_i$ and $Q_i=Q_{\sigma_i}(R_i)$. Since, for each $i\in I$, Q_i is a ring of quotients of R_i *w.r.t.* σ_i, $\prod_{i\in I} Q_i$ is a ring of quotients of R *w.r.t.* σ by the corollary to Theorem 3. Let $U=\prod_{i\in I} U_i\in\mathscr{F}$, and let $f\in \mathrm{Hom}_R(U,R)$. Then, for each $i\in I$, $\pi_i f\kappa_i\in\mathrm{Hom}_{R_i}(U_i, R_i)$. Thus there is a (unique) $q_i\in Q_i$ such that $\pi_i f\kappa_i(u_i)=u_iq_i$ for all $u_i\in U_i$ by [2, Theorem 3, Corollary]. Let $u\in U$ and $u(i)=u_i$, then $\kappa_i(u_i)=e_iu$ where e_i is an element of R such that $e_i(j)=\delta_{ij}$, the Kronecker's delta. Let $q\in\prod_{i\in I}Q_i$ be defined by $q(i)=q_i$ then $(uq)(i)=u(i)q(i)=u_iq_i=\pi_if\kappa_i(u_i)=\pi_if(e_iu)=\pi_i(e_i)f(u)=\pi_if(u)=f(u)(i)$ for all $i\in I$. Hence, for any $f\in\mathrm{Hom}_R(U,R)$, we have found a $q \in \prod_{i \in I} Q_i$ such that $f(u) = uq$. Therefore, by Lemma 5, $\prod_{i \in I} Q_{\sigma_i}(R_i)$ is isomorphic to $Q_\sigma(R) = Q_\sigma(\prod_{i \in I} R_i)$ as required.

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Kyungpook University
and
University of Wisconsin, Madison