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# FIXED POINTS OF GENERALIZED CONTRACTION MAPPINGS ON MENGER SPACES

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## 0. Introduction

Statistical metric spaces (briefly SM-spaces) were introduced by Menger [3], and Sehgal and Bharucha-Reid [7] introduced the notion of contraction mappings on Menger spaces and proved a fixed point theorem on such mappings.

Since metric spaces can be considered as SM-spaces, we may raise a question that a certain kind of fixed point theorems on metric spaces can be generalized for SM-spaces.

The aim of this paper is to give several fixed point theorems on Menger spaces which are generalizations of some wellknown fixed point theorems on metric spaces.

For this, in section 1 we introduce the notion of Menger spaces and their topological properties. In section 2 we prove various types of fixed point theorems for generalized contraction mappings on Menger spaces.

## 1. Basic definitions

Let $\mathbf{R}$ be the set of reals and $\mathbf{R}^+=\{x\in\mathbf{R}\mid x\geq 0\}$. A mapping $F:\mathbf{R}\to\mathbf{R}^+$ is said to be a distribution function if it is nondecreasing left-continuous with $\inf F=0$ and $\sup F=1$. The set of all distribution functions will be denoted by $L$.

DEFINITION 1.1. A statistical metric space (SM-space) is an ordered pair $(S, \mathcal{F})$, where $S$ is a (nonempty) set and $\mathcal{F}$ is a mapping of $S\times S$ into $L$ (we shall denote $\mathcal{F}(p, q)$ by $F_{pq}$) satisfying

(SM-I) $F_{pq}(x)=1$ for all $x>0$ if and only if $p=q$

(SM-II) $F_{pq}(0)=0$

(SM-III) $F_{pq}=F_{qp}$

(SM-IV) if $F_{pq}(x)=1$ and $F_{qr}(y)=1$, then $F_{pr}(x+y)=1$,

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for $p$, $q$, $r \in S$ and $x$, $y \in R$.

Definition 1.1 suggests that $F_{pq}(x)$ may be interpreted as the probability that the distance between $p$ and $q$ is less than $x$.

DEFINITION 1.2. A Menger space is a triple $(S, \mathcal{F}, \Delta)$, where $(S, \mathcal{F})$ is a SM-space and $\Delta : [0,1] \times [0,1] \to [0,1]$ is a mapping satisfying

$$(\Delta\text{-I}) \quad \Delta(a,1)=a, \quad \Delta(0,0)=0$$
$$(\Delta\text{-II}) \quad \Delta(a,b)=\Delta(b,a)$$
$$(\Delta\text{-III}) \quad \Delta(c,d) \geq \Delta(a,b) \text{ if } c \geq a \text{ and } d \geq b$$
$$(\Delta\text{-IV}) \quad \Delta(\Delta(a,b),c)=\Delta(a,\Delta(b,c))$$
$$(\Delta\text{-V}) \quad F_{pr}(x+y) \geq \Delta(F_{pq}(x), F_{qr}(y))$$

for all $p, q, r \in S$ and for all $x \geq 0$, $y \geq 0$.

A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ satisfying $(\Delta\text{-I})-(\Delta\text{-IV})$ is said to be a $\Delta$-norm.

The topology of a SM-space was introduced by Schweizer and Skla [5]. Let $p \in S$, and $\varepsilon, \lambda$ be positive reals. Then an $(\varepsilon, \lambda)$-neighborhood of $p$ is defined by

$$N_p(\varepsilon, \lambda) = \{q \in S \mid F_{pq}(\varepsilon) > 1-\lambda\}.$$

Due to [5] and [6], if $(S, \mathcal{F}, \Delta)$ is a Menger space, and $\Delta$ is continuous, then it is a Hausdorff space satisfying the first axiom of countability induced by the family $\{N_p(\varepsilon, \lambda) \mid p \in S, \ \varepsilon > 0, \lambda > 0\}$ of neighborhoods.

DEFINITION 1.3. Let $(S, \mathcal{F}, \Delta)$ be a Menger space. A sequence $\{p_n\}$ in $S$ is said to be fundamental in $S$ if for each $\varepsilon > 0$, $\lambda > 0$, there exists an integer $M(\varepsilon, \lambda)$, such that $F_{p_n p_m}(\varepsilon) > 1-\lambda$ whenever $n, m \geq M(\varepsilon, \lambda)$.

A Menger space $S$ is complete if each fundamental sequence in $S$ converges to an element in $S$.

The following theorem establishes a connection between metric spaces and Menger spaces.

THEOREM 1.4 [7]. *If $(S, d)$ is a metric space, then the metric $d$ induces a mapping $\mathcal{F} : S \times S \to L$, where $\mathcal{F}(p,q)$ $(p, q \in S)$ is defined by $\mathcal{F}(p,q)(x) = H(x-d(p,q))$, $x \in \mathbf{R}$, where $H(x)=0$ if $x \leq 0$ and $H(x)=1$ if $x > 0$. Further, if $\Delta : [0,1] \times [0,1] \to [0,1]$ is defined by $\Delta(a,b) = \min\{a,b\}$, then $(S, \mathcal{F}, \Delta)$ is a Menger space. It is complete if the metric $d$ is complete.*

The space $(S, \mathcal{F}, \Delta)$ so obtained is called the induced Menger space.

## 2. Fixed point theorems

We begin with the following fixed point theorem.

THEOREM 2. 1. *Let $(S, \mathcal{F}, \Delta)$ be a complete Menger space and $\Delta$ be continuous satisfying $\Delta(x, x) \geq x$, $x \in [0, 1]$.* Let $\phi : \mathbf{R} \to \mathbf{R}$ *be an upper semicontinuous mapping from the right such that $\phi(x) = 0$ if $x \leq 0$, and $0 < \phi(x) < x$ for $x > 0$.*

*Suppose that $T$ is a selfmapping of $S$ satisfying*

(1) $F_{TpTq}(\phi(x)) \geq \min\{F_{pq}(x), [F_{pTp}(x) + F_{qTq}(x)], [F_{pTq}(x) + F_{qTp}(x)]\}$

*for every $p, q$ in $S$, and for all $x \in \mathbf{R}$.*

*Then $T$ has a unique fixed point in $S$ and $\{T^n p_0\}$ converges to the fixed point of $T$ for each $p_0$ in $S$.*

*Proof.* First note that $\lim \phi^n(x) = 0$ for $x \in \mathbf{R}$. For an arbitrary point $p_0$ in $S$, put $p_n = T^n p_0$. Then we have by (1),

$$F_{p_{n+1}p_n}(\phi(x)) \geq \min\{F_{p_np_{n-1}}(x), [F_{p_np_{n+1}}(x) + F_{p_{n-1}p_n}(x)], [F_{p_np_n}(x) + F_{p_{n-1}p_{n+1}}(x)]\} = F_{p_np_{n-1}}(x),$$

so that for $x \in \mathbf{R}$ and $n = 1, 2, \cdots$,

$$F_{p_{n+1}p_n}(\phi^n(x)) \geq F_{p_0p_1}(x). \tag{2}$$

Now we claim that the sequence $\{p_n\}$ is fundamental. Suppose not. Then there exist $\varepsilon > 0$ and $\lambda > 0$, for any positive integer $l$, we can find $n, m$ with $n > m \geq l$ such that

$$F_{p_np_m}(\phi(\varepsilon)) < 1 - \lambda, \quad F_{p_{n-1}p_m}(\phi(\varepsilon)) \geq 1 - \lambda. \tag{3}$$

For this, we can choose $x > 0$ such that $F_{p_0p_1}(x) \geq 1 - \lambda$. Since $\lim \phi^n(x) = 0$, we may choose a positive integer $N$ with $\phi^N(x) \leq \phi(\varepsilon)$, so that

$$F_{p_np_{n+1}}(\phi(\varepsilon)) \geq F_{p_np_{n+1}}(\phi^n(x)) \geq F_{p_0p_1}(x) \geq 1 - \lambda$$

for all $n \geq N$ by (2). By (1), we have

$$F_{p_np_m}(\phi(\varepsilon)) \geq \min\{F_{p_{n-1}p_{m-1}}(\varepsilon), [F_{p_{n-1}p_n}(\varepsilon) + F_{p_{m-1}p_m}(\varepsilon)], [F_{p_{n-1}p_m}(\varepsilon) + F_{p_{m-1}p_n}(\varepsilon)]\}. \tag{4}$$

Suppose $l$ is sufficiently large such that $l \geq N+1$ and $\varepsilon - \phi(\varepsilon) \leq \phi^{l+1}(x)$. Then we have

$$F_{p_{n-1}p_n}(\varepsilon) \geq F_{p_{n-1}p_n}(\phi(\varepsilon)) \geq F_{p_{n-1}p_n}(\phi^{n-1}(x)) \geq F_{p_0p_1}(x) \geq 1 - \lambda$$

and by (3)

$$F_{p_{n-1}p_{m-1}}(\varepsilon)\geq\Delta(F_{p_{n-1}p_m}(\phi(\varepsilon)),F_{p_mp_{m-1}}(\varepsilon-\phi(\varepsilon))$$
$$\geq\Delta(1-\lambda,\ 1-\lambda)\geq1-\lambda.$$

Therefore by (4), $1-\lambda>1-\lambda$, which is a contradiction. Thus $\{p_n\}$ is fundamental. Since $(S,\mathcal{F},\Delta)$ is complete, $p_n$ converges to a limit $p$ in $S$.

Let $N_{Tp}(\varepsilon,\lambda)$ be any neighborhood of $Tp$ with $\varepsilon>0$, $\lambda>0$. Since $p_n\to p$, there exists an integer $N$ such that $p_n\in N_p(\varepsilon/2,\lambda/2)$ for all $n\geq N$. Then for $n\geq N$,

$$\begin{aligned}F_{p_{n+1}Tp}(\varepsilon)&\geq F_{Tp_nTp}(\phi(\varepsilon))\\&\geq\min\{F_{p_np}(\varepsilon),[F_{p_np_{n+1}}(\varepsilon)+F_{pTp}(\varepsilon)],\\&\qquad[F_{p_nTp}(\varepsilon)+F_{pp_{n+1}}(\varepsilon)]\}\\&>1-\lambda,\end{aligned}$$

since

$$\begin{aligned}F_{p_np_{n+1}}(\varepsilon)&\geq\Delta(F_{p_np}(\varepsilon/2),F_{pp_{n+1}}(\varepsilon/2))\\&\geq\Delta(1-\lambda/2,\ 1-\lambda/2)\geq1-\lambda/2>1-\lambda.\end{aligned}$$

Therefore $p_n\to Tp$. Since the space $(S,\mathcal{F},\Delta)$ is Hausdorff, $p=Tp$.

Now suppose that $q$ is another fixed point of $T$ in $S$. Then

$$\begin{aligned}F_{pq}(\phi(x))&=F_{TpTq}(\phi(x))\\&\geq\min\{F_{pq}(x),[F_{pp}(x)+F_{qq}(x)],[F_{pq}(x)+F_{pq}(x)]\}\\&=F_{pq}(x).\end{aligned}$$

for all $x\in\mathbf{R}$, so that $F_{pq}(\phi^n(x))\geq F_{pq}(x)$ for all $n=1,2,\cdots$. Since $\lim\phi^n(x)=0$, for any $\varepsilon>0$ and $\lambda>0$, there exist $x>0$ and an integer $n$ such that $F_{pq}(x)>1-\lambda$ and $\phi^n(x)\leq\varepsilon$. Therefore we have $F_{pq}(\varepsilon)>1-\lambda$, so that $F_{pq}(x)=1$ for any $x>0$. This shows that $p=q$.

As immediate corollaries, we have the followings.

THEOREM 2.2. *Let $(S,\mathcal{F},\Delta)$ and $\phi$ be as in* Theorem 2.1. *Suppose that $T$ is a selfmapping of $S$ satisfying*

$$F_{TpTq}(\phi(x))\geq F_{pq}(x)$$

*for every $p,q$ in $S$, and for all $x\in\mathbf{R}$. Then $T$ has a unique fixed point in $S$.*

COROLLARY 2.3. *Let $(S,\mathcal{F},\Delta)$ and $\phi$ be as in* Theorem 2.1. *Suppose that $T$ is a selfmapping of $S$ satisfying*

$$F_{T^npT^nq}(\phi(x))\geq F_{pq}(x)$$

*for some positive integer $n$, $p,q\in S$ and $x\in\mathbf{R}$. Then $T$ has a unique fixed point in $S$.*

In a complete metric space, Theorem 2.2 is that of Boyd and Wong [1].

Let $(S,\mathcal{F})$ be a SM-space, and $D$ be a subset of $S$. For $x\in\mathbf{R}$, we denote $F_D(x)$ by

$$F_D(x) = \inf\{F_{pq}(x) \mid p, q \in D\}.$$

A subset $D$ of $S$ is bounded if $\sup F_D = 1$.

Let $T$ be a selfmapping of $S$. For any $p, q$ in $S$, we will denote $D(p)$ and $D(p, q)$ by $D(p) = \{p, Tp, T^2p, \cdots\}$ and $D(p, q) = D(p) \cup D(q)$.

If the condition $\Delta(x, x) \geq x$ is omitted, we have another type of a fixed point theorem as follows.

THEOREM 2.4. *Let* $(S, \mathfrak{F}, \Delta)$ *and* $\phi$ *be as in* Theorem 2.1 *except the condition* $\Delta(x, x) \geq x$. *Suppose that* $T$ *is a continuous selfmapping of* $S$ *such that* $D(p)$ *is bounded for every* $p$ *in* $S$ *and*

$$F_{TpTq}(\phi(x)) \geq F_{D(p,q)}(x),$$

*for every* $p, q$ *in* $S$ *and* $x \in \mathbf{R}$.

*Then* $T$ *has a unique fixed point in* $S$ *and any iteration* $\{T^n p_0\}$ *converges to the fixed point of* $T$.

*Proof.* For an arbitrary $p_0$ in $S$ put $p_n = T^n p_0$. For any $x > 0$, $\{F_{D(p_n)}(x)\}$ is an increasing sequence, hence for some $0 \leq \lambda(x) \leq 1$, $F_{D(p_n)}(x) \to \lambda(x)$.

For $n \geq m \geq k$, $F_{D(p_n)}(x) \geq F_{D(p_n, p_m)}(x) \geq F_{D(p_k)}(x)$ implies

$$F_{p_{n+1}p_{m+1}}(\phi(x)) = F_{Tp_nTp_m}(\phi(x)) \geq F_{D(p_np_m)}(x) \geq F_{D(p_k)}(x).$$

Therefore we have $F_{D(p_{k+1})}(\phi(x)) \geq F_{D(p_k)}(x)$ for any $x > 0$.

Now suppose that $\lambda(x_0) < 1$ for some $x_0 > 0$. Then there exists $x > 0$ such that $F_{D(p_0)}(x) > \lambda(x_0)$, since $\sup F_{D(p_0)} = 1$.

Let $n$ be a positive integer with $\phi^n(x) \leq x_0$. Then $F_{D(p_n)}(x_0) \geq F_{D(p_n)}(\phi^n(x)) \geq F_{D(p_0)}(x) > \lambda(x_0)$, which is a contradiction. Hence we have $\lambda(x) = 1$ for all $x > 0$. Therefore the sequence $\{p_n\}$ is fundamental. Since $T$ is continuous and the space $(S, \mathfrak{F}, \Delta)$ is Hausdorff, there exists $p$ in $S$ such that $p_n \to p$ and $Tp = p$. Suppose that $q$ is another fixed point of $T$. Since $D(p, q) = \{p, q\}$, by the same way in the proof of Theorem 2.1, we have $p = q$.

In a complete metric space, Theorem 2.4 is a slight modification of Theorem 5 in [2]. But in [2], the continuity of $T$ is omitted. Variations of such a type fixed point theorem can be founded in [4].

In a compact Menger space, we have a more powerful fixed point theorem as follows.

THEOREM 2.5. *Let* $(S, \mathfrak{F}, \Delta)$ *be a compact Menger space and* $\Delta$ *be continuous. Let* $T$ *be a continuous selfmapping of* $S$ *satisfying the following contractive condition; for any* $p, q$ *in* $S$ *with* $p \neq q$ *and for any* $x > 0$, *there exists a real* $y > x$ *such that*

$$F_{TpTq}(x) \geq F_{D(p,q)}(y).$$

*Then* $T$ *has a unique fixed point in* $S$, *moreover any iteration* $\{T^n p\}$ *con-*

*verges to the fixed point of T.*

*Proof.* Since $T^{n+1}(S)\subset T^n(S)$, $n=1,2,\cdots$, by compactness of $S$ and continuity of $T$, $\{T^n(S)\}$ has the finite intersection property, so that $Y=\bigcap_{n=1}^{\infty}T^n(S)$ is a nonempty compact subset of $S$. We shall prove that $TY=Y$. Since $TY\subset Y$, let $p\in Y$. Then there exists $p_n\in T^n(S)$ such that $Tp_n=p$. By compactness of $S$, we may choose a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ which converges to some point $q$ in $S$. Since $p_{n_i}\to q$, $q\in T^{n_i}(S)$, so that $q\in Y$ and $Tq=p$.

Now we claim that $F_Y(x)=1$ for each $x>0$. Suppose $F_Y(x)=b<1$ for some $x>0$. Since $F_Y$ is increasing and lower semicontinuous, we may assume that $F_Y(x)<F_Y(y)$ for any $y>x$ by taking $x=\sup\{y\,|\,F_Y(y)=b\}$.

Since $Y$ is compact and $F_Y(x)=\inf\{F_{p,q}(x)\,|\,p,q\in Y\}$ and $\liminf F_{p_nq_n}(x)=F_{pq}(x)$ for $p_n\to p$ and $q_n\to q$, (see [5]), we can choose two distinct points $p, q$ in $Y$ such that $F_{p,q}(x)=b$.

Since $TY=Y$, there exists $p_1, q_1$ in $Y$ such that $Tp_1=p$, $Tq_1=q$. But by assumption, there exists $y>x$, such that

$$b=F_{pq}(x)=F_{Tp_1Tq_1}(x)\geq F_{D(p_1,q_1)}(y)\geq F_Y(y)$$

which is a contradiction.

Therefore $Y=\{p^*\}$ is a singleton. Then $T^np\to p^*$ for all $p$ in $S$ and $p^*$ is the unique fixed point of $T$.

For a compact metric space, we have the following theorem which is a generalization of Shin and Yeh [8].

THEOREM 2.6. *Let $(S,d)$ be a compact metric space and $T$ be a continuous selfmapping of $S$ satisfying*

$$d(Tp, Tq)<O(p,q)$$

*for $p,q$ in $S$ with $p\neq q$, where $O(p,q)$ is the diameter of $D(p,q)$. Then $T$ has a unique fixed point in $S$, and any iteration $\{T^np\}$ converges to the fixed point of $T$.*

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