J. Korean Math. Soc.
Vol. 19, No. 2, 1983

ANGULAR SEPARATIONS OF FINITE SETS IN E^2

BY E. P. MERKES*

1. Introduction

For an integer $n>1$, let K_n denote a set of n points in the Euclidean plane E^2. A partition of K_n is an unordered pair of nonempty subsets A and B of K_n such that $A \cup B = K_n$ and $A \cap B = \phi$. We denote such a partition by $[A, B]$ (or $[B, A]$). The number of partitions of K_n is clearly $2^{n-1}-1$.

For a given real $\theta, 0<\theta<\pi$, a partition $[A, B]$ of K_n is called a θ-separation, written (A, B), if there exists two lines that intersect in an angle of measure θ such that A and B respectively lie in the interior of the opposite vertical angles of measure θ determined by the pair of lines. We denote by $\eta(\theta, n)$ the maximum number of θ-separations over all sets K_n of n points in E^2. In particular, $\eta(\theta, 2)=1$ for all choices of θ. If $\theta>\frac{\pi}{3}$ and if the points of K_3 determine a triangle, each angle of which is less than θ, then there are three θ-separations of K_3. Hence, $\eta(\theta, 3)=3$ for $\theta>\frac{\pi}{3}$ since there are only three partitions of a set of three points in E^2.

A few years ago, the author and one of his students proved that $\eta\left(\frac{\pi}{2}, n\right)=n$ for $n>2$ [1]. The solution of this combinatorial problem was needed to determine the number of distinct domains of univalence for certain families of rational functions. The problem of finding $\eta(\theta, n)$ for other choices of θ does have implications in the theory of univalent functions although it appears to be an interesting and nontrivial problem itself. The method of proof in [1] can be easily extended to show $\eta(\theta, n)=n$ for $n>2$ when $\frac{\pi}{3}<\theta\leq\frac{\pi}{2}$. In this paper, we prove $\eta(\theta, n)=n-1$ when $0<\theta\leq\frac{\pi}{3}$. When $\theta=\pi$,

Received Seoptember 18, 1982.

*The research was sponsored by a grant from the Taft Committee, University of Cincinnati. The content of this paper was presented in lectures at Inha University (June 29, 1982), Soong Jeon University (June 30, 1982), Keimyung University (July 1, 1982) and Choong Buk University (July 3, 1982).

the two lines of the θ-separation coincide and $\eta(\pi, n)=\frac{n(n-1)}{2}$. The determination of $\eta(\theta, n)$ for $\frac{\pi}{2}<\theta<\pi$ is an open question.

More explicitly, we prove here the following result.

THEOREM. *For $0<\theta\leq\frac{\pi}{3}$, there are at most n-1 θ-separations of n distinct points in the Euclidean plane. For each $n>1$ there is a set of n distinct points in the plane such that there are exactly n-1 θ-separations.*

Our proof is by mathematical induction.

2. Preliminaries

A θ-separation for a set $K_3=\{k_1, k_2, k_3\}$ of three points in the plane has one point, k_1 say, in the interior of an angle of measure θ whereas the other two points are in the interior of the opposite vertical angle. Therefore the angle $\angle k_2k_1k_3$ at k_1 must have measure less than θ. If $0<\theta\leq\frac{\pi}{3}$, it follows that at least one of the other two angles in the triangle determined by k_1, k_2, and k_3 must exceed $\frac{\pi}{3}$ in measure. This implies that the vertex of this angle is a point that cannot be separated from the other two vertices by a θ-separation. We conclude $\eta(\theta, 3)\leq 2$. By selecting the points k_1, k_2, and k_3 such that two of the angles of the triangle determined by these three points each have measure less than θ, we prove $\eta(\theta, 3)\geq 2$ and, hence, $\eta(\theta, 3)=2$ when $0<\theta\leq\frac{\pi}{3}$. This can serve as the starting point of our induction.

Suppose $[A, B]$ is a partition of the set K_n of n points in the plane and $k\notin K_n$. The set $K_{n+1}=K_n\cup k$ (actually $K_n\cup\{k\}$ but the braces are dropped for simplicity of notation) has two partitions that naturally correspond to the partition $[A, B]$ of K_n, namely, $[A\cup k, B]$ and $[A, B\cup k]$. If either is a θ-separation of K_{n+1}, then by deleting the point k we conclude that the partition $[A, B]$ was a θ-separation of K_n. Hence, a θ-separation (A, B) of K_n corresponds to at most two θ-separations, $(A\cup k, B)$ and $(A, B\cup k)$, of K_{n+1}. The only other type of θ-separation of K_{n+1} that can arise is (k, K_n).

LEMMA 1. *Let $0<\theta\leq\frac{\pi}{3}$. If (k, K_n) is a θ-separation of $K_{n+1}=K_n\cup k$, $k\notin K_n$, then for any θ-separation (A, B) of K_n at most one of the two par-*

titions $[A \cup k, B]$, $[A, B \cup k]$ *is a* θ*-separation of* K_{n+1}.

Proof. Let $a \in A$ and $b \in B$, where (A, B) is a θ-separation of K_n. Since (k, K_n) is a θ-separation of K_{n+1}, the triangle determined by a, b, and k has an angle of measure less than θ at k. This implies the angle at a or at b of this triangle has measure exceeding θ and, hence, a or b cannot be separated by a θ-separation from the other two vertices of the triangle. We conclude that either $[A, B \cup k]$ or $[A \cup k, B]$ is not a θ-separation of K_{n+1}.

The next lemma is stated in a more general form than is necessary for the proof of our theorem. It is because this lemma cannot be further extended to the case when $\frac{\pi}{2} < \theta < \pi$ that the methods of proof in this paper and in [1] fail for choices of θ beyond $\frac{\pi}{2}$. In the lemma, we use the notation E-D for the complement of the set D in E, that is, for the set of all points of E that are not points of D.

LEMMA 2. *Let* $0 < \theta \leq \frac{\pi}{2}$. *If* (A, B) *is a* θ*-separation of* K_n *and if* A_1, B_1 *are respectively nonempty proper subsets of* A *and* B, *then the partition* $[A_1 \cup B_1, K_n - (A_1 \cup B_1)]$ *is not a* θ*-separation of* K_n.

Proof. Select a coordinate system such that one line of a θ-separation (A, B) of K_n is the horizontal (real) axis, the other line is in the first and third quadrants (or the vertical axis if $\theta = \frac{\pi}{2}$), and the origin is at the point of intersection of these lines. Then the only points in the first and third quadrant that are on a line which separates $A_1 \cup B_1$ and $K_n - (A_1 \cup B_1)$ into oppositive half-planes must be points in the interior of the vertical angles of measure θ of the θ-separation. The angle between two such lines, therefore, has measure less than θ. Hence, the partition $[A_1 \cup B_1, K_n - (A_1 \cup B_1)]$ cannot be a θ-separation of K_n.

LEMMA 3. *Let* $0 < \theta \leq \frac{\pi}{3}$. *If* $k \notin K_n$, *then there is at most one partition* $[A, B]$ *of* K_n *such that both* $(A \cup k, B)$ *and* $(A, B \cup k)$ *are* θ*-separations of* $K_n \cup k$.

Proof. If $[A, B]$ is a partition of K_n, then each other partition of K_n must have one of the following forms:

$$[A_1, K_n - A_1],\ [B_1, K_n - B_1],\ [A_1 \cup B_1, K_n - (A_1 \cup B_1)],$$

where A_1, B_1 are respectively proper nonempty subsets of A and B. Suppose

$(A,\ B\cup k)$ and $(A\cup k,\ B)$ are θ-separations of $K_{n+1}=K_n\cup k$. Hence, $(A,\ B)$ is a θ-separation of K_n. Now $[A_1\cup B_1,\ K_n-(A_1\cup B_1)]$ is not a θ-separation of K_n by Lemma 2. Therefore, adjuncting the point k to either of the sets in this partition cannot lead to a θ-separation of K_{n+1}. Since $(A,\ B\cup k)$ is a θ-separation of K_{n+1}, the partition $[A_1\cup k,\ K_n-A_1]$ cannot by Lemma 2 be a θ-separation of K_{n+1}. Indeed, points from the first set A of the θ-separation $(A, B\cup k)$ of K_{n+1} are transferred to the second set while a point of the second set, namely k, is transferred to the first set in building the partition $[A_1\cup k,\ K_n-A_1]$. Lemma 2 assures us that such a transformation does not produce θ-separations. Similarly $[A_1,\ (K_n-A_1)\cup k]=[A_1,\ (K_n\cup k)-A_1]$ is not a θ-separation of K_{n+1} since $(A\cup k,\ B)$ is a θ-separation. By symmetry what has been proved for A also applies when A is replaced by B. Thus, there is no second partition $[\tilde{A},\ \tilde{B}]$ such that $(\tilde{A}\cup k, \tilde{B})$ and $(\tilde{A},\ \tilde{B}\cup k)$ are θ-separations of K_{n+1}.

3. Proof of the Theorem.

Assume for some integer $n\geq 3$ that $\eta(\theta, n)\leq n-1$, where $0<\theta\leq\frac{\pi}{3}$. Let K_{n+1} be a set of $n+1$ points in the plane and let $k\in K_{n+1}$, $K_n=K_{n+1}-k$. The number of θ-separations of K_n is at most n-1. Each θ-separation of K_{n+1}, except (k, K_n) if it is a θ-separation, arises from the partitions $[A\cup k,\ B]$ or $[A,\ B\cup k]$, where (A, B) is a θ-separation of K_n. If (k, K_n) is a θ-separation of K_{n+1}, then by Lemma 1 at least one of the partitions $[A\cup k,\ B]$ or $[A,\ B\cup k]$ is not a θ-separation of K_{n+1}. Hence, the number of θ-separations of K_{n+1} is at most one greater than the number of θ-separations of K_n in this case. On the other hand, if $[k,\ K_n]$ is not a θ-separation of K_{n+1}, then there is at most one θ-separation, $(A,\ B)$ say, of K_n such that both $(A\cup k,\ B)$ and $(A,\ B\cup k)$ are θ-separations of K_{n+1} by Lemma 3. Again the number of θ-separations of K_{n+1} is at most one greater than those of K_n. It follows that $\eta(\theta,\ n+1)\leq n$. Since $\eta(\theta,\ 3)=2$, we have by induction $\eta(\theta,\ m)\leq m-1$ for all integers $m\geq 3$. (The inequality is also trivially true for $m=2$.)

It remains to prove $\eta(\theta,\ m)=m-1$ when $0<\theta\leq\frac{\pi}{3}$. This is accomplished by noting that the number of θ-separations of m points on a line is exactly m-1 for $m\geq 2$.

REMARK. If $\frac{\pi}{3}<\theta\leq\frac{\pi}{2}$, the proof in [1] can ensily be extended to establish the inequality $\eta(\theta,\ m)\leq m$ for $m>2$. To prove equality can hold, we determine the set K_m as follows. Select $m-2$ points in a coordinate plane of the form $(x, 0)$, where x is in the open interval $\cot\frac{\theta}{2}<x<\tan\theta$. The remaining two points are $(0, 1)$ and $(0, -1)$. The number of θ-separations of K_m in this case is exactly m.

4. Open Questions.

We have already mentioned that the value of $\eta(\theta, n)$ for $\frac{\pi}{2}<\theta<\pi$ is unknown. Of course, for $n>2$ we have $n\leq\eta(\theta, n)\leq\frac{n(n-1)}{2}$. We suspect the value of $\eta(\theta, n)$ for sufficiently large n changes at each θ of the form $\frac{(m-2)\pi}{m}$ $(m=3, 4, 5, \cdots)$, the measure of the angles of a regular polygon of m sides.

The beauty of the problem so far is that its resolution required only the most elementary mathematics. However, is there a shorter proof of the known results perhaps using techniques from the subject of "convexity"?

Finally, are there analogues of even the known results in Euclidean space E^d, $d>2$? Since we know of no application for this generalization, we have not attempted an extension to higher dimensions. Nonetheless the problem does appear to be of interest.

Reference

1. S. Lameier and E. P. Merkes, *Separation of points in the plane*, Math. Mag. **52** (1979), 108-110.

Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221
U. S. A.