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SOME PROBLEMS IN GROUP PRESENTATIONS

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1. Introduction

This paper has been motivated from two different sources. The first-named author visited the Republic of Korea in April 1982 when courses and seminars which he gave on combinatorial group theory revitalised interest in certain problems in group presentations. That these problems might now be more amenable was due to work of the second-named author in extending a suite of group theory programs based on programs originally developed in the Australian National University.

We shall consider problems which arise in the papers [1], [2] and [3]. In certain cases we are now able to give complete answers while in other cases we report on progress. In particular we give a new symmetrical presentation for the simple group $PSL(3,3)$ namely:

$$\langle a,b \mid a^{13}=1,\ a^4ba=b^3,\ b^4ab=a^3\rangle.$$

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We use the notation $[a, b]$ for $a^{-1}b^{-1}ab$ and C_n for the cyclic group of order n.

2. A group involving Fibonacci numbers

The class of groups motivating the work of [2] is

$$Y(n)=\langle a,b \mid (abab^{-1})^n=ba^{-1}bab^{-1}a,\ ab^2a^{-1}ba^2b^{-1}=1\rangle.$$

Using Theorem 5.8 and Theorem 6.1 of [2] we can write the final conjecture of that paper in the following form:

Conjecture. If $n\neq 0 \pmod 3$, $Y(n)$ is a finite group of order $4n(2n+3)g_n^3$ if n is even and $20n(2n+3)f_n^3$ if n is odd where g_n and f_n denote the nth terms of the Lucas and Fibonacci sequences respectively.

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In [2] the conjecture was proved for $n=-5, -4, -2, -1, 1, 2, 4$ while $Y(n)$ was proved infinite if $n\equiv 0 \pmod 3$. We have now verified that the conjecture is also true for $n=-10, -8, -7, 5, 7, 8, 10$.

The method in all these cases is the same. We first find a presentation for the subgroup $H=\langle a, b^2\rangle$ which has index $2n+3$ in $Y(n)$, using a machine implementation of the Reidemeister–Schreier and Tietze transformation methods. In all cases a two generator presentation for H is attained. Again using the machine implemented Reidemeister–Schreier and Tietze transformation methods we find a presentation for H', the derived group of H. In all the above cases H' is generated by two commutators and has index $4n$. The machine implementation of Todd–Coxeter finds the order of H' directly only in the cases $n=-7, -5$ and 7. We illustrate the types of presentations found in all the above seven cases and also illustrate the argument used to complete the $n=-10, -8, 8$ and 10 cases by looking in detail at $n=8$.

In this case the subgroup H for $Y(8)$ is found to have a presentation on x and y with relations

$$x^2y^{-1}x^{-1}yxy=1,$$
$$x^2y^5xy^8xyx^{-1}y^2x^{-1}y=1,$$
$$x^2y^3xyx^{-1}yx^{-1}y^{-1}x^{-1}y^2xyx^{-1}y^{-1}x^{-1}y^4xyx^{-1}y^3x^{-1}y=1,$$
$$x^2yxyx^{-1}y^{-1}x^{-1}y^2xyx^{-1}y^{-1}x^{-3}y^4xyx^{-1}y^{-1}x^{-1}y^2x^{-2}yx^2y^3x^{-1}y=1.$$

It may be worth remarking that the first of these relations is obtained in all seven cases and also that, for $Y(-8)$, a deficiency zero presentation for H is obtained, namely:

$$x^2y^{-1}x^{-1}yxy=1,$$
$$(x^{-1}y^3x^{-1}yx^2y^4x^{-1})^2y^{-1}=1.$$

Returning to the group $Y(8)$, $H'=\langle xyx^{-1}y^{-1}, x^{-1}y^{-1}xy\rangle$ and the presentation for H' obtained is on generators s and t with the eight relations

(1) $s^2ts^{-1}t^{-2}s^{-1}t=1,$

(2) $s^2t^{-1}s^{-1}t^2s^{-1}t^{-1}=1,$

(3) $sts^{-1}tst^{-1}s^{-1}t^{-1}=1,$

(4) $s^2t^2s^2t^{-2}s^{-2}t^2s^{-2}t^{-2}=1,$

(5) $s^7t^2s^4(tst^2s)^2t^3st^4stst^3st^5s^{-4}t^{-2}s^{-1}t^{-1}s^{-6}t^{-3}s^{-1}t^{-1}s^{-1}t^{-2}(s^{-1}t^{-3})^4$
$s^{-1}t^{-1}s^{-1}t^{-2}=1,$

(6) $s^7t^2s^4t^2s^5t^2s^4t^{-2}s^{-1}t^{-2}s^{-1}t^{-4}s^{-1}t^{-1}s^{-1}t^{-2}s^6t(s^3t^2s^4ts^3t^2s^3)^2st^{-3}s^{-1}$
$t^{-1}s^{-1}t^{-2}=1,$

(7) $(s^5s^2s^5t^2s^3t)^2s^3t(st^2st^2st^4st^2st^3)^2(st^2)^4sts^3t=1,$

(8) $s^7t^2s^5t^2s^4t^2(s^6t^2s^4ts^3t^2)^2s^5ts^3t^2s^4t^{-2}s^{-1}t^{-2}(s^{-1}t^{-3})^3ts^{-1}t^{-4}s^{-1}t^{-1}s^{-1}t^{-2}=1.$

The Todd-Coxeter easily finds that the index of

$$s^{-1}tst^{-1} \text{ in } H' \text{ is } 2209=47^2 \text{ and so } \langle s^{-1}tst^{-1}\rangle = H''.$$

But $s^{-1}tst^{-1}$ is central in H' for relation (3) gives

$$[t, s^{-1}ts]=1$$

while, using both relations (1) and (3) we see that

$$[s, tst^{-1}]=1.$$

But $H'/H'' \simeq C_{47}\times C_{47}$ so $s^{-1}tst^{-1}$ is an element of order dividing 47 since the above argument shows that it is contained in the Schur multiplier of $C_{47}\times C_{47}$, this Schur multiplier being C_{47}, see [5], Theorem 25. 11. A computer implementation of the nilpotent quotient algorithm shows that H' has a 47-quotient of order 47^3 and so H' has order 47^3. The order of $Y(8)$ is therefore $19\cdot 32\cdot 47^3=63, 124, 384$.

The largest order of the groups for which the conjecture has been verified is for the group $Y(10)$ and this order is 1, 711, 997, 640. The presentations for the appropriate subgroups of $Y(-10)$ and $Y(10)$ have considerably lengthier relations than the $Y(8)$ case illustrated above, but the method is identical.

3. A two generator two relation perfect group

In [3] we showed that

$$G=\langle a, b \mid a^2b^3=1,\ (ab)^7=(a^{-1}b^{-1}ab)^8a^2\rangle$$

is a finite perfect group of deficiency zero. G has the group $(2, 3, 7; 8)$, in the notation of Coxeter (see [4]) as a homomorphic image and Leech and Mennicke [6] have proved that $(2, 3, 7; 8)$ has order 10752. Also G has the simple group $PSL(2, 7)$ as a homomorphic image and is the only finite perfect group, other than the covering group of a finite simple group, known to have deficiency zero. Although we show in [3] that G is finite we were unable to determine its order We can now show that $|G|=43008 =4\cdot 10752$ in the following way.

Let $H=\langle a^{-1}b^{-1}ab,\ (ba^{-1}b^{-1}a)^2\rangle$. Then the Reidemeister-Schreier and Tietze programs show that H has index 7 in G and a presentation on generators x, y, z wth relations

$$x^2z^{-2}=1,$$
$$x^2y^3=1,$$

$$xyzxyz^{-1}=1$$
$$xy^{-1}z^{-1}x^{-1}y^{-1}zy^{-1}z^{-1}x^{-1}z^{-1}yx(zy)^2=1,$$
$$xyxzyz^{-1}y^2z^{-1}(xzy)^2z^{-1}y=1,$$
$$xyxy^{-1}xzxy^{-1}xyzyx^{-1}zxzy^{-1}z^{-1}x^{-1}z^{-1}=1,$$
$$(xy^{-1})^2z^{-1}x^{-1}y^{-1}zyz^{-1}(x^{-1}z^{-1})^3(xz)^2=1,$$
$$xy^{-1}zy^{-1}z^{-1}x^{-1}(y^{-1}z)^2yz^{-1}yxzyz^{-1}yx^{-1}z^{-1}=1.$$

Now take $K=\langle xy, (xz)^2\rangle$. Then K has index 16 in H and presentation on generators s, t with relations

$$t^8=1,$$
$$s^4tst^{-1}st=1,$$
$$s^2t^{-1}s^{-1}ts^{-1}t^3=1,$$
$$s^2t^{-1}s^{-1}t^{-3}s^{-1}t^{-1}=1,$$
$$st\ sts^{-1}t^{-2}s^{-1}t=1,$$
$$(st)^6=1,$$
$$(st^2s^{-1}t^2)^2=1,$$
$$s^2t^6=1.$$

The Todd–Coxeter coset enumeration algorithm shows easily that $|K|=384$ so the order of G is determined.

4. The groups $G(l, m, n)$

In [1] the groups

$$G(l, m, n)=\langle a, b \mid ab^m a^{-1}b^{-1}a^{-n}b^{1-l}=ba^m b^{-1}a^{-1}b^{-n}a^{1-l}=1\rangle$$

are discussed. Many problems are left unanswered with this class and here we report on some progress which can be made with the greater range of computational techniques now developed. While doing the work for [1] the following were natural questions to ask:

QUESTION 1. Are the groups $G(-1, -1, n)$ finite? If so, do they have order n^3 when $n\not\equiv -1 \pmod 5$ and order $120n^3$ when $n\equiv -1 \pmod 5$?

QUESTION 2. Are the groups $G(-2, n, -1)$, $n\geq 1$, finite? Which of the finite groups are non-metabelian?

QUESTION 3. Classify the groups $G(1-n, m, n)$ and the groups $G(1, m, n)$.

In [1] Question 1 is shown to be answered in the affirmative for $-3\leq n \leq 5$, $n\neq 0$. Despite considerable efforts we failed to determine $|G(-1, -1. -4)|$. We can now show that $|G(-1, -1, -4)|=64$.

Question 2 arises since $G(-2, n, -1)$ is finite for $1\leq n\leq 4$. The groups

$G(-2,2,-1)$ and $G(-2,3,-1)$ have orders 1320 and 5832 and they are discussed in detail in [1] where it is shown that they are not metabelian. However we can now show that $G(-2,5,-1)$ is infinite. The derived group G' of $G(-2,5,-1)$ has index 48 and may be generated by

$$[a,b],\ [a^{-1},b],\ [a^{-1},b^{-1}],\ ab^7,\ ba^7,\ [a^{-1},b^{-2}],\ [a^{-2},b^{-1}].$$

The presentation obtained for G' is very complicated but we are able to show that $G'/G'' \cong ZZ*ZZ*H$ for some finite group H. Hence G is infinite.

Regarding Question 3 there appears in [1] a table for the group $G(l,m,n)$ with $-4 \leq m,\ n \leq 5$. This table is complete except for $G(1,5,5)$, $G(1,-4,-4)$, $G(1,3,5)$ and $G(1,-4,-2)$. The first of these groups we can now show to have order 72, the second is infinite (having G/G' infinite). The remaining two groups are easily shown to be isomorphic and the relation $(ab)^3=1$ is readily deduced. Hence

$$G=G(1,3,5)=\langle a,\ b \mid a^4ba=b^3,\ b^4ab=a^3,\ (ab)^3=1\rangle.$$

Our collection of programs allow us to show $|G/G'|=21$, $|G'/G''|=169$ and that the 13-quotient of G' is 13^3. Also the index of $\langle a,\ b^{13}\rangle$ in G is 432. This suggests looking at the homomorphic image $\bar{G}$ where

$$\bar{G}=\langle a,\ b \mid a^{13}=1,\ a^4ba=b^3,\ b^4ab=a^3\rangle.$$

Now $\bar{G}$ is a perfect group of order 5616 so must be $PSL(3,3)$. This is a particularly elegant presentation for $PSL(3,3)$ being both symmetrical and concise. We have been unable to determine whether $G(1,3,5)$ is finite but the above information shows that its order is at least $2^4 \cdot 3^3 \cdot 7 \cdot 13^3$.

The information contained in [1] suffices to deal with all the groups $G(1-n,m,n)$ for $-4 \leq m,\ n \leq 5$ except $G(-2,-4,3)$, $G(3,-4,-2)$, $G(-1,-4,2)$ and $G(4,-4,-3)$. The first two of these groups are infinite. The derived groups of both have C_2*C_2 as a homomorphic image. The remaining two groups have orders 55 and 605, respectively.

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