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# PROPERTIES OF REGULAR HOMOMORPHISMS

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In this paper, we will show a relation between regular homomorphisms and proximal homomorphisms of minimal sets, and give another necessary and sufficient conditions for a homomorphism of minimal sets to be regular.

A *transformation group* $(X, T)$ wll consist of a jointly continuous action of the topological group $T$ on the compact Hausdorff space $X$. The group $T$, with identity $e$, is assumed to be topologically discrete and will remain fixed through this paper, so we may write $X$ instead of $(X, T)$. A transformation group is said to be *minimal* if every point has dense orbit, or, equivalently, if it contains no proper closed invariant subset. Minimal transformation groups are also referred to as minimal sets. Every transformation group contains minimal sets. A *homomorphism* from $(X, T)$ to $(Y, T)$ is a continuous map $\pi : X \to Y$ such that $\pi(xt)=\pi(x)t$, $(x \in X,\ t \in T)$. If $Y$ is minimal, $\pi$ is always onto. Especially, a homomorphism from $(X, T)$ into itself is called an *endomorphism* of $(X, T)$.

A point $x \in X$ is said to be *almost periodic* if, given any neighborhood $U$ of $x$, the set $A=\{t \in T : xt \in U\}$ is syndetic, i. e., there exists a compact set $K \subset T$ such that $AK=T$. A point is almost periodic if and only if its orbit closure is minimal.

The compact Hausdorff space $X$ carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X \times X$. Two points $x, x' \in X$ are said to be *proximal* if, given any index $\alpha$, there exists a $t \in T$ such that $(x, x')t \in \alpha$. The set of proximal pairs in $X$ is called the *proximal relation* and denoted by $P(X)$. $X$ is said to be *distal* if the proximal relation equals the diagonal and is said to be *proximal* if the proximal relation equals $X \times X$.

A homomorphism $\pi : X \to Y$ determines a closed, invariant, equivalence relation on $X$, called $R(\pi)$ where

$$R(\pi)=\{(x, x') \in X \times X : \pi(x)=\pi(x')\}.$$

Conversely, a closed, invariant equivalence relation $R$ on $X$ determines an epimorphism $\pi : X \to X/R$. Given $\pi : X \to Y$ we define the *relative (to $\pi$) pro-*

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*ximal relation* on $X$ to be the intersection of $R(\pi)$ with the proximal relation on $X$. The homomorphism $\pi: X \to Y$ is said to be *distal* if the relative proximal relation equals the diagonal and is said to be *proximal* if the relative proximal relation equals $R(\pi)$.

Regular minimal sets were first studied by Auslander in [1]. A minimal set is said to be *regular minimal* if it is isomorpic to a minimal right ideal in some enveloping semigroup, or, equivalently, if for any two point $x$, $x'$ in $X$, there is an endomorphism $\theta$ of $X$ such that $\theta(x)$, $x'$ are proximal. In [8], the auther extended these notions to general transformation groups (not necessarily minimal).

Shoenfeld extended these notions to homomorphisms in [7] as follows: A homomorphism $\pi: X \to Y$ of minimal sets is *regular* if for $x$, $x'$ in $X$ with $\pi(x)=\pi(x')$, there exists an endomorphism $\theta$ of $X$ such that $\theta(x)$, $x'$ are proximal and $\pi\theta=\pi$, or, equivalently, if for any two points $x, x' \in X$ with $(x, x')$ almost periodic and $\pi(x)=\pi(x')$, there exists an endomorphism $\theta$ of $X$ such that $\theta(x)=x'$ and $\pi\theta=\pi$.

The following theorem will show a relation between regular homomorphisms and proximal homomorphisms of minimal sets.

THEOREM 2. 1. *Let $\pi: X \to Y$ be a homomorphism of minimal sets. Then $\pi$ is a proximal homomorphism if and only if it is a regular homomorphism and the only endomorphism $\theta$ of $X$ such that $\pi\theta=\pi$ is the identity.*

*Proof.* Let $\pi$ be a proximal homomorphism. It is clear that $\pi$ is regular. Let $\varphi_i$ be an endomorphism of $X$ with $\pi\varphi_i=\pi (i=1, 2)$. Since $1\times\varphi_i: X\times X \to X\times X$ is a homomorphism and $\Delta_X=\{(x, x)\,|\,x\in X\}$ is a minimal set, $W_i=(1\times\varphi_i)(\Delta_X)$ is a minimal subset of $X\times X$. Since $W_i \subset R(\pi)$, $W_i$ is also a minimal subset of $R(\pi)$. Let $\pi: R(\pi)\to Y$ be a homomorpnism given by $\pi(x, x')=\pi(x)=\pi(x')$. Then we will show that $\pi$ is a proximal homomorphism. Let $\pi(x, x')=\pi(y, y')$. Then we have $\pi(x)=\pi(x')=\pi(y)=\pi(y')$. Since $\pi$ is a proximal homomorphism, there exists $p\in E(X)$ such that $xp=yp$. Since $\pi(x'p)=\pi(x')p=\pi(y')p=\pi(y'p)$, there exists $q\in E(X)$ such that $(x'p)q=(y'p)q$. Consequently there exists $pq\in E(X)$ such that $(x, x')pq=(y, y')pq$. This means $(x, x')$, $(y, y')$ are proximal. Thus $\pi$ is a proximal homomorphism. Since $Y$ is a minimal set, there exists a unique minimal set in $R(\pi)$. (See p. 17 in [6]). Thus we have $W_1=W_2$. This means $\varphi_1=\varphi_2$. Since the identity $1_X$ satisfies $\pi\cdot 1_X=\pi$, we have $\varphi_1=\varphi_2=1_X$.

Conversely let $\pi(x)=\pi(x')$. Since $\pi$ is a regular homomorphism, there exists endomorphism $\theta$ of $X$ such that $(\theta(x), x')\in P(X, T)$ and $\pi\theta=\pi$. But the only endomorphism $\theta$ of $X$ such that $\pi\theta=\pi$ is the identity. Thus $\theta$ is the identity. Therefore $(x, x')=(1_X(x), x')=(\theta(x), x')\in P(X, T)$.

COROLLARY 2.2. *Let $X$ be a minimal set. Then $X$ is proximal if and only if it is regular minimal and the only endomorphism of $X$ is the identity.*

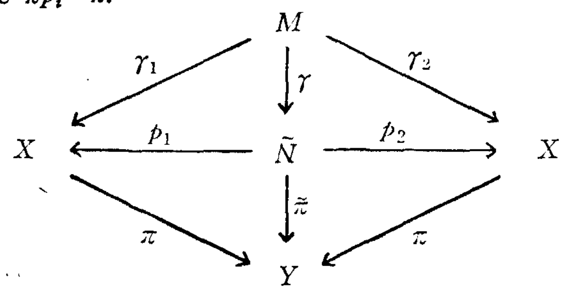
Corollary 2.2 was proved by the auther in [8].

Let $\pi: X\to Y$ be a homomorphism of minimal sets and $\tilde{N}$ be a minimal subset of $X\times X$ such that $\tilde{N}\subset R(\pi)$. Then the homomorphism $\tilde{\pi}: \tilde{N}\to Y$ given by $\tilde{\pi}(x, x')=\pi(x)=\pi(x')$ is said to be a *minimal homomorphism* of $\pi$ in $X\times X$. A homomorphism $\pi: X\to Y$ is said to be *coalescent* if every endomorphism $\theta$ of $X$ with $\pi\theta=\pi$ is an automorphism.

In [7] Shoenfeld showed that regular homomorphisms are coalescent. The following theorem is the converse of Shoenfeld's result and give an another necessary and sufficient condition for a homomorphism to be regular.

THEOREM 2.3. *Let $\pi: X\to Y$ be a homomorphism of minimal sets. Then $\pi$ is a regular homomorphism if and only if $\pi$ is coalescent and every minimal homomorphism of $\pi$ in $X\times X$ is isomorphic to $\pi$.*

*Proof.* Only if: That $\pi$ is coalescent was proved by Shoenfeld. Let $\tilde{\pi}: \tilde{N}\to Y$ be any minimal homomorphism of $\pi$ in $X\times X$ and $M$ be a universal minimal set. Since $\tilde{N}$ is a minimal set, there exists a homomorphism $\gamma: M\to\tilde{N}$. If we consider the projection $p_i: \tilde{N}\to X$ given by $p_i(x_1, x_2)=x_i$, then we have $\pi p_i=\tilde{\pi}$.




We will show that $p_1$ is an isomorphism. Since $X$ is minimal, it is sufficient to show that $p_1$ is a monomorphism. Suppose $p_1$ is not a monomorphism, then there exist two distinct points $(x, y)$, $(x, y')$ in $\tilde{N}$ such that $y\neq y'$. Let $\gamma_1=p_1\gamma$ and $\gamma_2=p_2\gamma$. Since $\pi\gamma_1=\pi\gamma_2$ and $\pi$ is a regular homomorphism, there is an automomorphism $\theta$ of $X$ such that $\pi\theta=\pi$ and $\gamma_1=\theta\gamma_2$ by Proposition 2.2.8 in [7]. Since $\gamma$ is an epimorphism, there exist $m, m'\in M$ such that $\gamma(m)=(x, y)$, $\gamma(m')=(x, y')$. Then we know $\gamma_1(m)=\gamma_1(m')$. Since $\theta$ is an automorphism and $\gamma_2(m)\neq\gamma_2(m')$, we have $\gamma_1(m)=\theta\gamma_2(m)\neq\theta\gamma_2(m')=\gamma_1(m')$. This contradicts. Thus $\tilde{\pi}$ is isomorphic to $\pi$.

If: Let $\pi(x)=\pi(x')$ and $(x, x')$ be an almost periodic point of $X\times X$. We must show that there exists an automorphism $\theta$ of $X$ such that $\theta(x)=x'$ and $\pi\theta=\pi$. Since $(x, x')$ is an almost periodic point in $X\times X$, $\tilde{N}=\overline{(x, x')T}$ is a minimal subset in $X\times X$ such that $\tilde{N}\subset R(\pi)$. Then the homomorphism $\tilde{\pi}: \tilde{N}\to Y$ is a minimal homomorphism of $\pi$ in $X\times X$. Thus $\tilde{\pi}$ is isomorphic to $\pi$ and there exists an isomophism $\varphi: X\to\tilde{N}$ such that $\tilde{\pi}\varphi=\tilde{\pi}$. Consider the projection $p_i: \tilde{N}\to X$. Then $\theta_i=p_i\varphi$ satisfies $\pi\theta_i=\pi(p_i\varphi)=\tilde{\pi}\varphi=\pi$. Since $\pi$ is coalescent, $\theta_i$ is an automorphism. Using $\theta_i=p_i\varphi$ and $\varphi$ are isomorphisms, we have $p_i=\theta_i\varphi^{-1}$ is also an isomorphism. If we take $\theta=p_2p_1^{-1}$, then $\theta$ is an automorphism of $X$ and $\theta(x)=p_2p_1^{-1}(x)=p_2(x, x')=x'$.

COROLLARY 2.4. *Let $X$ be a minimal set. Then $X$ is regular minimal if and only if it is coalescent and it is isomorphic with every minimal set $M$ contained in $X\times X$.*

Corollary 2.4 was proved by Auslander in [1].

THEOREM 2.5. *Let $\pi: X\to Y$ be a homomorphism of minimal sets. Then $\pi$ is proximal if and only if every endomorphism $\theta$ of $X$ such that $\pi\theta=\pi$ is the identity and any minimal homomorphism of $\pi$ in $X\times X$ is isomorphic to $\pi$.*

*Proof.* By Theorem 2.1 and Theorem 2.3.

Let $M$ be a universal minimal transformation group and $G$ be the group of automorphisms of $(M, T)$. In [2], Auslander showed that a homomorphism $\gamma$ from $M$ to a minimal set $X$ determine a subgroup $G(X,\gamma)=\{\alpha\in G: \gamma\alpha=\gamma\}$ of $G$. Shoenfeld showed that if $\pi: X\to Y$ is a homomorphism of minimal sets and $\gamma: M\to X$ is a homomorphism, then $G(X,\gamma)$ is a normal subgroup of $G(Y, \pi\gamma)$.

The following theorem is a partial converse of the above result.

THEOREA 2.6. *Let $X, Y$ be minimal sets, $\gamma: M\to X$ be a homomorphism and $\pi: X\to Y$ be a distal homomorphism. Then $G(X, \gamma)$ is a normal subgroup of $G(Y, \pi\gamma)$ if and only if $\pi$ is regular.*

*Proof.* Let $\pi(x)=\pi(x')$. Since $\gamma: M\to X$ is an epimorphism, there exist $m, m'$ such that $\gamma(m)=x$, $\gamma(m')=x'$. Since $M$ is minimal and $m\in M$, there exists an idempotent $u$ in a minimal right ideal $I$ of $E(M)$ such that $mu=m$. Now $\pi\gamma(m')=\pi\gamma(m)=\pi\gamma(mu)=\pi\gamma(m)u=\pi\gamma(m')u=\pi\gamma(m'u)$. Thus $\gamma(m')$, $\gamma(m'u)$ belong to the same fiber of $\pi$. Since $\gamma(m'u)=\gamma(m')u$. we have $(\gamma(m'), \gamma(m'u))\in P(X, T)$. If we use the fact that $\pi$ is distal, then we get

$$\gamma(m')=\gamma(m'u) \tag{1}$$

Since $M$ is a universal minimal set, $M$ is regular. Hence $\pi\gamma$ is a regular homomorphism. Therefore there exists an automorphism $\alpha\in G(Y, \pi\gamma)$ such that $(\alpha m, m')\in P(M, T)$. This implies that there exists a minimal right ideal $K$ of $E(M)$ such that $(\alpha m)p=m'p(p\in K)$. Since $u\in I$ there exists an idempotent $u'\in K$ such that $uu'=u$, $u'u=u'$. Thus $\alpha m=\alpha mu=\alpha muu'=\alpha mu'=m'u'$.

Now $$\gamma(\alpha m)=\gamma(m'u')=\gamma(m')u'=\gamma(m'u)u'=\gamma(m'uu')$$
$$=\gamma(m'u)=\gamma(m') \text{ by (1).}$$

Thus we have

$$\gamma(\alpha m)=\gamma(m')=x' \tag{2}$$

If we define $\bar{\alpha}: X\to X$ *by* $\bar{\alpha}(x)=\gamma\alpha\gamma^{-1}(x)$, then $\bar{\alpha}$ is well defined. For let $\gamma(m_1)=\gamma(m)=x$. Since $\gamma: M\to X$ is a regular homomorphism, there exists $\beta\in G(X, \gamma)$ such that $(\beta m, m_1)\in P(M, T)$. Since $\alpha: M\to M$ is an automorphism, we have $(\alpha\beta m, \alpha m_1)\in P(M, T)$. Now $\gamma: M\to X$ is an epimorphism, so $(\gamma\alpha\beta m, \gamma\alpha m_1)\in P(X, T)$. By the hypothesis, $G(X, \gamma)$ is a normal subgroup of $G(Y, \pi\gamma)$. Since $\alpha\in G(Y, \pi\gamma)$ and $\beta\in G(X, \gamma)$, we have $\alpha\beta=\beta^*\alpha$ for some $\beta^*\in G(X, \gamma)$. Therefore we have

$$\gamma\alpha\beta m=\gamma\beta^*\alpha m=\gamma\alpha m \tag{3}$$

Now $\pi(\gamma\alpha m_1)=\pi\gamma(m_1)=\pi\gamma(m)=\pi\gamma\alpha(m)=\pi(\gamma\alpha\beta m)$. Since $\pi$ is distal and $(\gamma\alpha\beta m, \gamma\alpha m_1)\in P(X, T)$, we have

$$\gamma\alpha\beta m=\gamma\alpha m_1 \tag{4}$$

By (3), (4), we get $\gamma\alpha(m)=\gamma\alpha(m_1)$. Thus $\bar{\alpha}$ is well defined. It is trivial $\bar{\alpha}$ is continuous. Now, $\bar{\alpha}(xt)=\gamma\alpha(mt)=\gamma\alpha(m)t=x't=\bar{\alpha}(x)t$ by (2). Hence $\bar{\alpha}$ is an endomorphism. Since $\bar{\alpha}(x)=x'$, we obtain $(\bar{\alpha}(x), x')\in P(X, T)$. Thus $\pi$ is a regular homomorphism.

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