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ON THE UNIFORM INTEGRABILITY OF CONTINUOUS PARAMETER STOCHASTIC PROCESSES

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J. L. Doob (1975) [6] introduced a notion of optionally separable processes which generizes both the separable processes and well-measurable processes.

G. Johnson and L. L. Helms [8] showed that right continuous supermartingale $(X_t)_{0\leq t\leq\infty}$ is of class (D) if and only if $\lim_n E(X_{T_n})=E(X_\infty)$ for every increasing sequence (T_n) of stopping times converging to $+\infty$ (also see [10, p. 102]). In this paper we will extend G. Johnson and L. L. Helms' result to the optionally separable processes.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $(\mathcal{F}_t)_{0\leq t\leq\infty}$ an increasing right continuous family of sub-σ-algebra. $\mathcal{F}_0$ includes all of the null sets. A process (X_t) is adapted if X_t is $\mathcal{F}_t$-measurable for each t. Unless some other convention is stated explicitly, process (X_t) means a stochastic process $(X_t)_{0\leq t\leq\infty}$ adapted to $(\mathcal{F}_t)$. A function $T:\Omega\rightarrow R_+\cup\{+\infty\}$ is a stopping time for $(\mathcal{F}_t)$ iff $\{T\leq t\}\in\mathcal{F}_t$ for all $t\in R_+$. It is known that $\mathcal{F}_T=\{F\in\mathcal{F}: F\cap\{T\leq t\}\in\mathcal{F}_t$ for all $t\in R_+\}$ is a σ-algebra. In order that X_T is $\mathcal{F}_T$-measurable, it suffices to assume that the process (X_t) is progressive i. e., for all $t\in R_+$ the map $[0,t]\times\Omega\rightarrow\bar{R}$ defined by $(s,\omega)\rightarrow X_s(\omega)$ is measurable with respect to $\mathcal{B}[0,t]\times\mathcal{F}_t$.

J. L. Doob introduced optionally separable processes [6].

Definition [6]. If (X_t) is a process, a sequence (S_n) of finite stopping times is called an optional separability set for (X_t) if for each ω, the set $(S_n(\omega): n\in N)$ contains 0 and is dense in $[0,\infty)$ and the graph of the sample function $t\rightarrow X_t(\omega)$ is in the closure of the graph restricted to the countable dense set $(S_n(\omega): n\in N)$.

Note that the set $(S_n\wedge k: n, k\geqslant 1)$ is also an optionally separability set whose stopping times are bounded. If the stopping times S_n can be chosen constant, then the process is separable. J. L. Doob [6] has shown that every well measurable process is (indistinguishable from) an optionally separable process.

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Definition [9]. Let $(X_t)_{0\le t\le\infty}$ be a progressive process. (X_t) is called regular supermartingale if for any stopping times S and T with $S\le T$, we have $E(X_T^-)<\infty$ and $E(X_T|\mathcal{F}_S)\le X_S$.

It is known that a right continuous supermartingale is regular supermartingale [10, p. 98].

We begin with a "maximal" lemma which is known for separable (or "right continuous") supermartingale [7, p. 353].

LEMMA 1. *Let* $(X_t)_{t\in[0,\infty]}$ be a progressive, optionally separable regular supermartingale. Then for each $\lambda>0$,

(a) $\lambda P\{\inf\limits_{0\le t\le\infty} X_t(\omega)<\lambda\}\geqslant\int_{\{\inf\limits_{0\le t\le\infty} X_t(\omega)<\lambda\}} X_\infty$

(b) $\lambda P\{\sup\limits_{0\le t\le\infty} X_t(\omega)>\lambda\}\le\int_{\{\sup\limits_{0\le t\le\infty} X_t>\lambda\}} X_\infty-EX_\infty+EX_0$

Proof. Let (S_n) be an optionaly separability set such that the S_n are bounded. For fixed n let $(S_1', S_2', \cdots, S_n')$ be the rearrangement of $(S_1, S_2, \cdots, S_n)$ in increasing order, then $(S_1', S_2', \cdots, S_n')$ are also stopping times (see [1, 6]).

Let $\Lambda_1=\{X_{S_1'}(\omega)<\lambda\}$

$$\Lambda_k=\{X_{S_j'}(\omega)\geqslant\lambda,\ 1\le j<k,\ X_{S_k'}<\lambda\},\ 1<k\le n$$

$$\Lambda_\infty=X_{S_j'}(\omega)\geqslant\lambda,\ 1\le j\le n,\ X_\infty<\lambda\}$$

Then $\Lambda_1, \cdots, \Lambda_n, \Lambda_\infty$ are disjoint, $\Lambda_k\in\mathcal{F}_{S_k'}$, $\Lambda_\infty\in\mathcal{F}_\infty$ and

$$\begin{aligned}\Lambda_1\cup\cdots\cup\Lambda_n\cup\Lambda_\infty&=\{\min_{j=1,\cdots,n,\infty} X_{S_j'}(\omega)<\lambda\}\\&=\{\min_{j=1,\cdots,n,\infty} X_{S_j}(\omega)<\lambda\},\ \text{where } S_\infty=\infty\end{aligned}$$

Using the supermartingale inequality and the fact $X_{S_k'}(\omega)\le\lambda$ on Λ_k, we find that

$$\begin{aligned}\int_{\{\min\limits_{j=1,\cdots,n,} X_{S_j}<\lambda\}} X_\infty&=\sum_{k=1,\cdots,n,\infty}\int_{\Lambda_k} X_\infty\le\sum_{k=1,\cdots,n,\infty}\int_{\Lambda_k} X_{S_k'}\\&\le\lambda\sum_k P(\Lambda_k)=\lambda P\min_{j=1,\cdots,n,} X_{S_j}<\lambda\})\end{aligned}$$

As $n\to+\infty$, $\{\min\limits_{j=1,\cdots,} X_{S_j}<\lambda\}$ increase to $\{\inf\limits_{j=1,\cdots,\infty} X_{S_j}(\omega)<\lambda\}=\{\inf\limits_{0\le t\le\infty} X_t(\omega)<\lambda\}$ by the definition of optionally separability set. By taking $n\to\infty$, we get $\int_{\{\inf\limits_{0\le t\le\infty} X_t<\lambda\}} X_\infty\le\lambda P(\{\inf\limits_{0\le t\le\infty} X_t<\lambda\})$

(b) As in (a) let $S'_1, \cdots, S'_n$ be a rearrangement of $S_1, S_2, \cdots, S_n$ in increasing order. Let $S_0'=0$ and $S_\infty'=+\infty$. Since (X_t) is a regular supermartingale, $(X_{S_0'}, X_{S_1'}, \cdots, X_{S_n'}, X_{S_\infty'})$ is a supermartingale.

Let $M_n=\{\max_{j=0,1\cdots,n} X_{S'_j}>\lambda\}$. Define

$$\tau(\omega)=\begin{cases}\min\{S'_j : X_{S'_j}>\lambda\} & \text{if } \omega\in M_n\\ \infty & \text{if } \omega\notin M_n\end{cases}$$

Then τ is a stopping time for $(\mathfrak{F}_{S'_0}, \mathfrak{F}_{S'_1}, \cdots, \mathfrak{F}_{S'_n}, \mathfrak{F}_{S'_\infty})$. By the optional sampling theorem, we have $E(X_{S'_0})\geq E(X_\tau)$, so that

$$E(X_{S'_0})\geq E(X_\tau)=\int_{M_n}X_\tau+\int_{\Omega\setminus M_n}X_\infty$$
$$\geq \lambda P(M_n)+E(X_\infty)-\int_{M_n}X_\infty$$

Thus we have

$$\lambda P(M_n)\leq\int_{M_n}X_\infty-E(X_\infty)+E(X_0)$$

Since M_n increase to $\{\sup_{1\leq j\leq\infty}X_{S'_j}>\lambda\}=\{\sup_{0\leq t\leq\infty}X_t(\omega)>\lambda\}$, by taking limit as $n\to\infty$, we obtain

$$\lambda P(\{\sup_{0\leq t\leq\infty}X_t>\lambda\})\leq\int_{\{\sup_{0\leq t\leq\infty}X_t>\lambda\}}X_\infty-E(X_\infty)+E(X_0).$$

THEOREM 2. *Let $(X_t)_{0\leq t\leq\infty}$ be a progressive, optionally separable process such that for any stopping time S there exists a stopping time $T\geq S$ with $EZ_T^-<+\infty$. Then there is the smallest regular supermartingale $(Z_t)_{0\leq t\leq\infty}$ satisfying $X_t\leq Z_t$ for all t.. Furthermore $(Z_t)_{0\leq t\leq\infty}$ is progressive, optionally separable process and $Z_T=\operatorname{ess\,sup}_{S\geq T} E(X_S|\mathfrak{F}_T)$ for any stopping time T. $(Z_t)_{0\leq t\leq\infty}$ is called the Snell envelope of $(X_t)_{0\leq t\leq\infty}$.*

Proof. The proof of this theorem is similar to the proof in the case of well-measurable progress [9] and is omitted.

THEOREM 3. *Let $(X_t)_{t\in[0,\infty]}$ be a progressive, optionally separable process. Assume that $\sup_T|E(X_T)|<+\infty$, where supremum is taken over the set of all extended stopping times. If $\lim E(X_{T_n})=E(X_\infty)$ for every increasing sequence (T_n) of extended stopping times converging to ∞, then (X_t) is of class (D), i.e., (X_T) is uniformly integrable over the set of all extended stopping times.*

REMARK. G. Johnson and L. L. Helms [8] showed that a non-negative right continuous supermartingale $(X_t)_{t\in[0,\infty]}$ is of class (D) if $\lim E(X_{T_n})=E(X_\infty)$ for every increasing sequence (T_n) of extended stopping times converging to ∞. So Theorem 3 is an extension of this result.

Proof. Let $(Z_t)_{t\in[0,\infty]}$ be the Snell envelope of $(X_t)_{t\in[0,\infty]}$ in the Theorem 2. Define

$$R_n=\inf\{t\ ;\ Z_t(\omega)\geq n\}.\quad (\inf\phi=\infty)$$

The (R_n) is an increasing sequence of extended stopping times converging *to* ∞, since $\sup_t Z_t < +\infty$ a.e. by Lemma 1. We will break the remainder of the proof into three steps.

Step 1. We will show that $E(Z_{R_n}) \to E(X_\infty)$. Suppose that $\lim E(Z_{R_n}) > EX_\infty + \varepsilon$ for some $\varepsilon > 0$. Since $E(Z_{R_n}) = \sup_{T \geq R_n} E(X_T) > E(X_\infty) + \varepsilon$ for each n, there exists an stopping time $S_n \geq R_n$ such that $E(X_{S_n}) > E(X_\infty) + \varepsilon$. It remains to show that we can replace the sequence (S_n) by an increasing sequence. Define

$$T_n'(\omega) = \min\{S_k(\omega) \mid S_k(\omega) \geq R_n(\omega),\ X_{S_k}(\omega) \geq E(X_{S_n} \mid \mathfrak{F}_{S_k})\}$$

Then $R_n \leq T_n' \leq S_n$, $X_{T_n'}(\omega) \geq E(X_{S_n} \mid \mathfrak{F}_{T_n'})$ and T_n' is an extended simple stopping time, because

$$\{\omega\,;\, S_k(\omega) \geq R_n,\ X_{S_k} \geq E(X_{S_n} \mid \mathfrak{F}_{S_k})\} \in \mathfrak{F}_{S_k}.$$ Let $T_n = \max_{1 \leq j \leq n} T_j'$.

Then (T_n) is an increasing sequence of extended simple stopping times converging *to* ∞. Now we will show that $E(X_{T_n}) \geq E(X_{T_n'}) \geq E(X_{S_n}) \geq E(X_\infty) + \varepsilon$, which contradicts the hypothesis of theorem. For fixed n, let $T_1'' = T'_n$, $T_{i+1}'' = T_i \vee T_i'$, then $T_n' = T_1' \leq T_2'' \cdots \leq T_{n-1}'' \leq T_n'' = T_n$. We assert that $E(X_{T_i''}) \leq E(X_{T_{i+1}''})$ for all i, thus we have $E(X_{T_n'}) \leq E(X_{T_n})$.

$$\begin{aligned}
E(X_{T_i''}) &= \int_{T_i'' = T_{i+1}''} X_{T_{i+1}''} dP + \int_{T_i'' < T_i'} X_{T_i''} dP \\
&\leq \int_{T_i'' = T_{i+1}''} X_{T_{i+1}''} dP + \int_{T_i'' < T'} E(X_{S_i} \mid \mathfrak{F}_{T_i''}) dP \\
&= \int_{T_i'' = T_{i+1}''} X_{T_{i+1}''} dP + \int_{T_i'' < T_i'} X_{S_i} dP \\
&= \int_{T_i'' = T_{i+1}''} X_{T_{i+1}''} dP + \int_{T_i'' < T_i'} E(X_{S_i} \mid \mathfrak{F}_{T_i'}) dP \\
&\leq \int_{T_i'' = T_{i+1}''} X_{T_{i+1}''} dP + \int_{T_i'' < T_i'} X_{T_i'} dP \\
&= \int_{T_i''' = T_{i+1}''} X_{T_{i+1}''} dP + \int_{T_i'' = T_{i+1}} X_{T_{i+1}''} dP = E(X_{T_{i+1}''})
\end{aligned}$$

The first and second inequalities followed from the definition of the stopping time T_i'.

Step 2. We will prove that $(Z_t)_{t \in [0, \infty]}$ is of class (D). Let T be an arbitrary stopping time and T' stopping time defined by

$$T'(\omega) = \begin{cases} T(\omega) & \text{if } Z_T(\omega) \geq n \\ +\infty & \text{otherwise} \end{cases}$$

Then we have $R_n \leq T'$ and consequently $E(Z_{R_n}) \geq E(Z_{T'}) \geq E(Z_\infty)$.
By step 1 we obtain that

$$\int_{\{Z_T\geq n\}} Z_T+\int_{\{Z_T<n\}} Z_\infty\to E(Z_\infty) \quad \text{as } n\to\infty$$

On the other hand, $\{\sup_{0\leq t\leq\infty} Z_t\leq n\}\subset\{Z_T\leq n\}$ for any stopping time T. Since $\sup_{0\leq t\leq\infty} Z_t<+\infty$ a.e. by Lemma 1, $P(\{\sup_{0\leq t\leq\infty} Z_t\leq n\})\to 1$ as $n\to\infty$. Therefore $P(\{Z_T\leq n\})\to 1$ as $n\to\infty$ uniformly on T. Thus

$$\int_{\{Z_T\geqslant n\}} Z_T=\int_{\{Z_T\geqslant n\}} Z_T{}^+\to 0 \quad \text{as } n\to\infty$$

uniformly in T, which implies $(Z_T{}^+)_{T\in\Gamma}$ is uniformly integrable where Γ is the set of all extended stopping times. From the relation $E(Z_\infty|\mathfrak{F}_T)\leq Z_T$, we derive uniform integrability of $(Z_T{}^-)_{T\in\Gamma}$.

Step 3. We will show that $(X_T)_{T\in\Gamma}$ is uniformly integrable. Since $X_t\leq Z_t$ for all t and $(Z_T{}^+)_{T\in\Gamma}$ is uniformly integrable, it follows that $(X_T{}^+)_{T\in\Gamma}$ is uniformly integrable. In order to prove that $(X_T{}^-)_{T\in\Gamma}$ is uniformly integrable, consider process $(-X_t)_{t\in[0,\infty]}$ which satisfies all the conditions of the theorem. Using $(-X_T)^+=X_T{}^-$ and step 2 we obtain that $(X_T{}^-)_{T\in\Gamma}$ is uniformly integrable. Thus $(X_T)_{T\in\Gamma}$ is uniformly integrable.

COROLLARY 4. *Let $(X_t)_{t\in R_+}$ be progressive, optionally separable process with* $\sup|E(X_T)|<+\infty$. *If $E(X_T)_n\to E(X_T)$ for every increasing sequence (T_n) of finite stopping times, which converges to any finite stopping time T, then $(X_{t\wedge T})_{t\in R_+}$ is of class (D) for any finite stopping time T.*

Proof. For any finite stopping time T, let $Y_t=X_{t\wedge T}$ for $t\in[0,\infty]$, then $(Y_t)_{t\in[0,\infty]}$ is progressive, optionally separable process. Applying Theorem 3 to the process (Y_t), we obtain that (Y_t) is of class (D).

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