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OSCILLATORY PROPERTIES FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH DAMPED TERM

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1. Introduction

Consider the following nonlinear second order differential equation with damped term

(1) $$y''(t)+p(t)y'(t)+q(t)f(y(t))=0$$

and

(2) $$y''(t)+p(t)y'(t)+q(t)f(y(g(t)))=0,$$

where $p, q\in C[t_0, \infty)$, $f\in C(R)$, $yf(y)>0$ for $y\neq 0$, $g'(t)\geqq 0$ for $t\geqq t_0$ and $\lim_{t\to\infty} g(t)=\infty$. We define $r(t)\equiv\exp(\int_{t_0}^{t} p(s)ds)$. We restrict our attention to solutions $y(t)$ of (1) which exist on some half-line $[t_0, \infty)$ and are nontrivial for all large t. A solution $y(t)$ of (1) is called oscillatory if $y(t)$ has zeros for arbitrarily large t, otherwise, a solution is said to be nonoscillatory. Equation (1) is oscillatory if all solutions of (1) are oscillatory. Recently, Yeh [8] proved some oscillatory results for equation (1) by using Kamenev's [4] method. Many author's have studied equation (1) (see [1, 3, 5-7]). In this paper, we propose another simple but useful oscillation criterion for equations (1) and (2). Especially, our results can be applied to all examples of Yeh [8], and also to the Emden-Fouler equation and the Fermi-Thomas equation.

2. Oscillation theorems

THEOREM 1. *Let $f'(y)$ exist and $f'(y)>0$ for $y\in\mathbf{R}'\equiv\mathbf{R}-\{0\}$. If*

(3) $$\int_{t_0}^{\infty} r(t)q(t)dt=\infty \text{ and } \int_{t_0}^{\infty}\frac{dt}{r(t)}=\infty,$$

then equation (1) *is oscillatory.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $y(t)>0$ for $t\geqq t_1\geqq t_0$, since a parallel argument holds when $y(t)<0$. Multiplying (1) by $r(t)/f(y(t))$ and integr-

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ating from t_1 to t, we obtain

(4) $$\int_{t_1}^{t} \frac{r(s)y''(s)}{f(y(s))} ds + \int_{t_1}^{t} \frac{r(s)p(s)y'(s)}{f(y(s))} ds + \int_{t_1}^{t} r(s)q(s)ds = 0.$$

By using $r'(t) = r(t)p(t)$ and (4), we have

(5) $$\frac{y'(t)r(t)}{f(y(t))} - C + \int_{t_1}^{t} \frac{r(s)f'(y(s))(y'(s))^2}{(f(y(s)))^2} ds + \int_{t_1}^{t} r(s)q(s)ds = 0,$$

whence we obtain

(6) $$\frac{y'(t)r(t)}{f(y(t))} \leqq C - \int_{t_1}^{t} r(s)q(s)ds,$$

where C is a constant.

By (3) and (6), we can obtain

(7) $$y'(t) < 0 \quad \text{for } t \geqq t_2 \geqq t_1.$$

From (5) with using (3) and (7), it follows that there is $t_3 \geqq t_2$ such that

(8) $$3 + \int_{t_3}^{t} \frac{r(s)f'(y(s))(y'(s))^2}{(f(y(s)))^2} \mathrm{ds} \leqq \frac{r(t)[-y'(t)]}{f(y(t))}.$$

Multiplying (8) by

$$\frac{f'(y(t))[-y'(t)]}{f(y(t))} \{3 + \int_{t_3}^{t} \frac{r(s)f'(y(s))(y'(s))^2}{(f(y(s)))^2} \mathrm{ds}\}^{-1} \geqq 0$$

and integrating from t_3 to t, we have

(9) $$\log \frac{f(y(t_3))}{f(y(t))} \leqq \log \{3 + \int_{t_3}^{t} \frac{r(s)f'(y(s))(y'(s))^2}{(f(y(s)))^2} \mathrm{ds}\}.$$

By (8) and (9), we obtain

(10) $$f(y(t_3)) \leqq r(t)[-y'(t)] \quad \text{for } t \geqq t_3.$$

Dividing (10) through by $r(t)$ and integrating from t_3 to t, we have

$$y(t) \leqq y(t_3) - f(y(t_3)) \int_{t_3}^{t} \frac{1}{r(s)} ds, \quad t \geqq t_3,$$

which, because it is supposed that $y(t) > 0$ for $t \geqq t_1$, contradicts (3).

Q. E. D.

REMARK. Theorem 1 corresponds to Theorem 1 of Yeh [8] and applies to all Examples of Yeh [8].

EXAMPLE 1[8]. Consider the equation

(11) $$y''(t) + \frac{1}{t} y'(t) + \frac{1}{t^2} y(t) = 0, \quad t \geqq 1.$$

Since $r(t) = \exp(\int_1^t \frac{1}{s} ds) = \exp(\log t) = t$, all conditions of Theorem 1 are satisfied. Hence, equation (11) is oscillatory.

EXAMPLE 2[8]. Consider the equation

(12) $$y''(t)+\frac{1}{2t}y'(t)+\frac{1}{4t}y(t)=0, \quad t\geqq 1.$$

Since $r(t)=\sqrt{t}$, all conditions of Theorem 1 are satisfied. Hence, equation (12) is oscillatory.

EXAMPLE 3. Consider the equation

(13) $$y''(t)+(\sin t)y'(t)+(1-\cos t)y(t)=0, \quad t\geqq \frac{\pi}{2}.$$

Since $r(t)=\exp(\int_{\pi/2}^{t}\sin s\,\mathrm{ds})=\exp(-\cos t)$, all conditions of Theorem 1 are satisfied. Hence, equation (13) is oscillatory. In fact, $y(t)=\sin t$ is a solution of (13).

REMARK. Theorem 1 is easier to apply to Example 3 rather than Theorems 1 and 2 of Yeh [8].

THEOREM 2. *Let* $q(t)\geqq 0$ *and* $f(y)/y\geqq k>0$ *for* $y\neq 0$. *If* (3) *holds, then equation* (1) *is oscillatory.*

Proof. Assume that $y(t)$ is a nonoscillatory solution of (1). Multiplying (1) by $r(t)/y(t)$ and integrating from t_1 to t, where t_1 is so chosen that $y(t)>0$ for $t>t_1$, we obtain

(14) $$\int_{t_1}^{t}\frac{r(s)y''(s)}{y(s)}\mathrm{ds}+\int_{t_1}^{t}\frac{r(s)p(s)y'(s)}{y(s)}\mathrm{ds}+\int_{t_1}^{t}kr(s)q(s)\mathrm{ds}\leqq 0.$$

By (14) with using $r'(t)=r(s)p(s)$, we have

(15) $$\frac{y'(t)r(t)}{y(t)}-C+\int_{t_1}^{t}\frac{r(s)(y'(s))^2}{(y(s))^2}\mathrm{ds}+k\int_{t_1}^{t}r(s)q(s)\mathrm{ds}\leqq 0,$$

where C is a constant.

By (3) and (15), we obtain

(16) $$y'(t)<0 \quad \text{for} \quad t\geqq t_2\geqq t_1.$$

From (15) with using (3) and (16), it follows that there is a $t_3\geqq t_2$ such that

(17) $$3+\int_{t_3}^{t}\frac{r(s)(y'(s))^2}{(y(s))^2}\mathrm{ds}\leqq\frac{r(t)[-y'(t)]}{y(t)}.$$

Multiplying (17) by

$$\frac{[-y'(t)]}{y(t)}\{3+\int_{t_3}^{t}\frac{r(s)(y'(s))^2}{(y(s))^2}\mathrm{ds}\}^{-1}$$

and integrating from t_3 to t, we have

(18) $$\log\frac{y(t_3)}{y(t)}\leqq\log\{3+\int_{t_3}^{t}\frac{r(s)(y'(s))^2}{(y(s))^2}\mathrm{ds}\}.$$

By (17) and (18), we obtain

$$y(t_3)\leqq r(t)[-y'(t)], \quad t\geqq t_3.$$

The rest of the proof proceeds as Theorem 1. Q. E. D.

REMARK. Theorem 2 corresponds to Theorem 2 of Yeh [8] and applies to Examples 1–3.

3. The case with deviating argument

THEOREM 3. *Let $f'(y)$ exist, $f'(y)>0$ for $y\in \mathbf{R}'\equiv\mathbf{R}-\{0\}$ and $q(t)\geqq 0$. If* (3) *holds, then equation* (2) *is oscillatory.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (2). Without loss of generality, we may assume that $y(t)>0$ for $t\geqq t_1\geqq t_0$, since a parallel argument holds when $y(t)<0$. Multiplying (2) by $r(t)/f(y(g(t)))$ and integrating from t_2 to t, where t_2 being so large that $g(t)\geqq t_1$, we obtain

$$(19)\qquad \int_{t_2}^{t}\frac{r(s)y''(s)}{f(y(g(s)))}ds+\int_{t_2}^{t}\frac{r(s)p(s)y'(s)}{f(y(g(s)))}ds+\int_{t_2}^{t}r(s)q(s)ds=0.$$

By (19), $f'(y)>0$ and $g'(t)\geqq 0$, we have

$$(20)\qquad \frac{y'(t)r(t)}{f(y(g(t)))}\leqq C-\int_{t_2}^{t}r(s)q(s)ds,$$

where C is a constant.

From (3) and (20), we obtain

$$(21)\qquad y'(t)<0 \quad \text{for} \quad t\geqq t_3\geqq t_2.$$

On the other hand, from (2), we may write

$$(22)\qquad r(t)y''(t)+r(t)p(t)y'(t)+r(t)q(t)f(y(g(t)))=0.$$

By integrating (22) from $t_4(>t_3)$ to t, we have

$$r(t)y'(t)\leqq r(t_4)y'(t_4).$$

By dividing this and integrating t_4 to t, we obtain

$$y(t)-y(t_4)\leqq r(t_4)y'(t_4)\int_{t_4}^{t}\frac{ds}{r(s)},$$

which, (21) and (3) lead a contradiction as $t\to\infty$. Q. E. D.

EXAMPLE 4. Consider the equation

$$(23)\qquad y''(t)+(\sin t)y'(t)+(1-\cos t)y(t+2\pi)=0, \quad t\geqq\frac{\pi}{2}.$$

Equation (23) is oscillatory by Theorem 3. In fact, $y(t)=\sin t$ is an oscillatory solution of (23).

4. Application to the Emden–Fouler equation

The Emden–Fouler equation [2, 7] encountered in astrophysics and Fermi–Thomas equation in atomic physics take the following form

$$(24)\qquad (t^{p}y'(t))'+t^{\lambda}y(t)^{\gamma}=0, \quad t\geqq 1,$$

where p, λ, γ positive constants, γ the ratio of odd integers. Equation (24) is written in the form

$$y''(t)+\frac{p}{t}y'(t)+t^{\lambda-p}y(t)^{\gamma}=0. \tag{25}$$

Bobisud [2] proved that equation (25) was oscillatory provided $\gamma \leqq 1$ and $\lambda \geqq p-1$. Wong [7] studied equation (25) for $p>1$. Here, we can show that Theorem 1 covers the other case, for example, $\gamma=1$, $p=1$ and $\lambda=-(1/2)$. Consider the equation (25) for $\gamma=1$, $p=1$ and $\lambda=-(1/2)$. Since $r(t)=\exp(\int_1^t \frac{1}{s}ds)=t$, all conditions of Theorem 1 are satisfied. Hence, equation (25) is oscillatory, and also equation (24) is oscillatory.

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