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CONVERSE MEASURABILITY THEOREM FOR GAUSSIAN PROCESSES

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let (Ω, d) be a complete metric space. Let D be a closed interval $[a, b]$ on the real line $\mathbf{R}^1$. Consider a Gaussian process $X(\cdot, \cdot)$ defined on $(\Omega, \mathcal{F}, P)$ and D, and let $t=(t_1, \cdots, t_n)$, where $a=t_0<t_1<\cdots<t_n\leqq b$. Let X_t be a random vector $(X(t_1, \cdot), \cdots, X(t_n, \cdot))$ determined by the given Gaussian process $X(\cdot, \cdot)$ and t, Then X_t is a measurable transformation from $(\Omega, \mathcal{F})$ into $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, where $\mathbf{R}^n$ is the n-dimensional Euclidean space, and $\mathcal{B}(\mathbf{R}^n)$ is the σ-algebra of Borel sets in $\mathbf{R}^n$.

Now we define a probability measure μ determined by X_t on the σ-algebra $\mathcal{B}(\mathbf{R}^n)$ as follows;

$$\mu(B)=P(X_t^{-1}(B)) \text{ for every } B \text{ in } \mathcal{B}(\mathbf{R}^n).$$

F. Koehler showed that E is a Lebesgue measurable set if and only if $X_t^{-1}(E)$ is a Wiener measurable set for the Wiener process $X(\cdot, \cdot)$ (unpublished). D. L. Skoug obtained Koehler's result for the Yeh-Wiener process. For a discussion of the Wiener measure space, the Wiener process, the Yeh-Wiener measure space and the Yeh-Wiener process see [2], [3], [6], [7] and [8].

In Section 2 we introduce the basic concepts and well-known results.

In Section 3 we show that if E is a Lebesgue measurable set in $\mathbf{R}^n$, then $X_t^{-1}(E)$ is in $\mathcal{F}$ for any Gaussian process (see [Theorem 3-1]). But it is not known whether $X_t^{-1}(E)\in\mathcal{F}$ implies that E is a Lebesgue measurable set in $\mathbf{R}^n$. In this paper we investigate that under what conditions on the Gaussian process $X(\cdot, \cdot)$ the above statement is true.

2. Preliminaries

In this section we will collect some basic concepts and well-known results

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and describe them in a convenient form for our purposes.

DEFINITION 2.1. Let P be a probability measure on the Borel σ-algebra $\mathcal{B}(\Omega)$ for a given metric space (Ω, d). The probability space $(\Omega, \mathcal{B}(\Omega), P)$ is called the probabilistic metric space.

PROPOSITION 2.2. *If A is a Borel set in a probabilistic metric space Ω, then for every $\varepsilon>0$, there exists an open subset G of Ω such that $A\subset G$ and $P(G-A)<\varepsilon$.*

Proof. See [4, Theorem 1.2, p. 27].

DEFINITION 2.3. Let $D=[a, b]$ be a closed interval on $\mathbf{R}^1$ and $a=t_0<t_1<\cdots<t_n=b$, where $t_j=a+\frac{j}{2^n}(b-a)$ for $j=0, 1, 2, \cdots, 2^n$. Define $\prod^k=\{t_{j\cdot 2^k} : t_{j\cdot 2^k}=a+\frac{j}{2^{n-k}}(b-a),\ j=0,1,2,\cdots, 2^{n-k}\}$ for $k=0,1,2,\cdots, n$. Then it is obvious that $\prod^n\subset\prod^{n-1}\subset\cdots\subset\prod^0$. By a primary binary interval $I(k\,;j)$, we mean an interval of the form $I(k\,;j)=[t_{(j-1)\cdot 2^k}, t_{j\cdot 2^k}]$.

The following proposition is taken from [1] without proof.

PROPOSITION 2.4. *Let f be a real valued function defined on $\prod^0$. Suppose that f is such a function that if s and t are the endpoints of some primary binary interval, then*

$$|f(s)-f(t)|\leqq gh|t-s|^r, \text{ where } h>0,\ g=\frac{1}{2}(1-2^{-r}) \text{ and } 0<r<\frac{1}{2}.$$

Then for any pair s and t on $\prod^0$, we have

$$|f(s)-f(t)|\leqq h|t-s|^r.$$

PROPOSITION 2.5. *Let $X(\cdot, \cdot)$ be a Gaussian process on a probability space $(S, \mathcal{S}, \nu)$ and an interval $D=[a, b]\subset\mathbf{R}_1$.*
If (1) *mean $E[X(t, \cdot)]=0$ for every t in D.*
(2) *variance $V[X(t, \cdot)-X(s, \cdot)]\leqq B|t-s|^\beta$ for every s and t in D, where B, $\beta>0$ are the given constant numbers, then for every s, t $(s\neq t)$ in D,*

$$\nu\{w\in S : |X(t, w)-X(s, w)|>h|t-s|^r\} < \frac{1}{h}\left(\frac{2B}{\pi}\right)^{\frac{1}{2}}|b-a|^{(\frac{1}{2}\beta-r)}\exp\left\{-\frac{h^2}{2B|t-s|^{\beta-2r}}\right\}$$

where $0<r<\frac{1}{2}\beta$ and $h>0$.

Proof. See [7, Theorem 21.2, p. 308].

3. Converse Measurability Theorem for Gaussian Processes

THEOREM 3. 1. *Let $(S, \mathcal{S}^*, \nu)$ be a complete probability space and let $X(\cdot, \cdot)$ be a Gaussian process on $(S, \mathcal{S}^*, \nu)$ and an interval $D=[a, b]\subset R^1$. Define*

$$J_t(E)=X_t^{-1}(E)=\{w\in S : (X(t_1, w), \cdots, X(t_n, w))\in E\}$$

where $E\subset\mathbf{R}^n$, $t=(t_1, \cdots, t_n)\in\mathbf{R}^n$ with $a=t_0<t_1<\cdots<t_n\leqq b$ and $X_t=(X(t_1, \cdot), \cdots, X(t_n, \cdot))$ is the n-dimensional random vector. Let g_t be the density function of X_t. Then if E is a Lebesgue measurable set in $\mathbf{R}^n$, then $J_t(E)$ is in $\mathcal{S}^*$, and

$$\nu(J_t(E))=\int_E g_t(u)\ dm(u)$$

where $u=(u_1, \cdots, u_n)\in\mathbf{R}^n$ and m is the Lebesgue measure on $\mathbf{R}^n$.

Proof. Case 1 : Let B be in the Borel σ-algebra $\mathcal{B}(\mathbf{R}^n)$ of $\mathbf{R}^n$. Then $J_t(B)$ is in $\mathcal{S}^*$, because $J_t(B)=X_t^{-1}(B)$ and Xt is a measurable transformation from $(S, \mathcal{S}^*)$ into $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$. Hence we have

$$\nu(J_t(B))=\nu\{w\in S : (X(t_1, w), \cdots, X(t_n, w))\in B\}$$
$$=\int_B g_t(u)\ dm(u).$$

Moreover, if $m(B)=0$, then

(3. 1) $$\nu(J_t(B))=0.$$

Case 2: Let E be any Lebesgue measurable set in $\mathbf{R}^n$. Then $E=B\cup N$, where B is a Borel set in $\mathbf{R}^n$. and N is a subset of a Borel null set N_0. But $J_t(N)\subset J_t(N_0)$ and $\nu(J_t(N_0))=0$ by (3. 1). Hence $J_t(N)$ is in $\mathcal{S}^*$ since $(S, \mathcal{S}^*, \nu)$ is complete. Therefore we have

$$J_t(E)=J_t(B)\cup J_t(N) \text{ is in } \mathcal{S}^* \text{ and}$$
$$\nu(J_t(E))=\nu(J_t(B))=\int_B g_t(u)dm(u)$$
$$=\int_{B\cup N} g_t(u)dm(u)=\int_E g_t(u)dm(u).$$

This completes the proof of the Theorem.

REMARK. Let $(\Omega, \mathcal{B}(\Omega), P)$ be a probabilistic complete metric space and let $(\Omega, \mathcal{F}, P)$ be the completion of $(\Omega, \mathcal{B}(\Omega), P)$. F. Koehler proved the converse of Theorem 3-1 for the Wiener process, which is called Koehler's Theorem, and D. L. Skoug [6] proved Koehler's Theorem for the Yeh-Wiener process. It is not known whether Koehler's Theorem holds for all Gaussian processes. We will prove this converse theorem for certain Gaussian processes on $(\Omega, \mathcal{F}, P)$ and $D=[a, b]\subset\mathbf{R}^1$. First of all, we state it and introduce some definitions and four lemmas which are followed immediately by the proof.

THEOREM 3.2. *(Koehler's Theorem for Gaussian processess.) Let* $(\Omega, \mathcal{B}(\Omega), P)$ *be a probabilistic complete metric space with a metric* d_1. *Let* $(\Omega, \mathcal{F}, P)$ *be the completion of* $(\Omega, \mathcal{B}(\Omega), P)$. *Assume that* $X(\cdot, \cdot)$ *is a continuous Gaussian process on* Ω *and* $D=[a, b]\subset \mathbf{R}^1$ *such that it satisfies the following conditions;*

(1) $X(t, \cdot)$ *is continuous on* Ω *for every* t *in* D.

(2) $X(a, w)=0$ *for every* w *in* Ω.

(3) $d_1(w_1, w_2)\leqq c\, d_2(X(\cdot, w_1), X(\cdot, w_2))$ *for some* $c>0$, *where* d_2 *is the metric induced by the sup-norm in the function space* $C(D)$ *of all real valued continuous functions on* D.

(4) *mean* $E(X(t, \cdot))=0$ *for every* t *in* D, *and for every* s, t $(s\neq t)$ *in* D, *variance* $V(X(t, \cdot)-X(s, \cdot))\leqq B|t-s|^{\beta}$, *where* B *and* β *are positive constant numbers. For any* $E\subset \mathbf{R}^n$, *define*

$$K=J_t(E)=\{w\in\Omega : (X(t_1, w), \cdots, X(t_n, w))\in E\}.$$

Then if K *is in* $\mathcal{F}$, *then* E *is a Lebesgue measurable set in* $\mathbf{R}^n$.

DEFINITIONS 3.3. Let $0<r<\frac{1}{2}\min(\beta, 1)$, where $\beta>0$, and let $X(\cdot, \cdot)$ be a continuous Gaussian process satisfying the conditions (1), (2), (3), and (4) as in Theorem 3.2. Let $C_0(D)$ be the space of all real valued continuous functions x on $D=[a, b]\subset\mathbf{R}^1$ with $x(a)=0$. Let $h>0$ be a given constant number. Define $A_h=\{x\in C_0(D) : |x(t)-x(s)|\leqq h|t-s|^r, \forall t, s\in D\}$ $C_h=\{w\in\Omega : |X(t, w)-X(s, w)|\leqq h|t-s|^r, \forall t, s\in D\}$ $B_h=\{X(\cdot, w) : w\in C_h\}$ =the collection of sample functions for the Gaussian process $X(\cdot, \cdot)$ as in Theorem 3.2 for every w in C_h.

REMARK. Note that B_h is contained in A_h for every $h>0$ by the continuity of $X(\cdot, w)$ on D for all $w\in\Omega$. It is well known that A_h is sequentially compact with respect to the uniform topology of $C_0(D)$ [1].

LEMMA 3.4. *The subset* C_h *of* Ω *is sequentially compact in a complete metric space* (Ω, d_1).

Proof. First of all, let us show that C_h is closed with respect to the metric topology of (Ω, d_1). Let $\{w_n\}$ be a sequence in C_h which converges to w in Ω. By the continuity of $X(t, \cdot)$, $t\in D$, and the fact that w_n is in C_h, $n=1, 2, 3, \cdots$, we have

$$\begin{aligned}(3.2)\quad |X(t, w)-X(s, w)| &\leqq |X(t, w)-X(t, w_n)|+|X(t, w_n)-X(s, w_n)| \\ &\quad +|X(s, w_n)-X(s, w)| \\ &<\frac{\varepsilon}{2}+h|t-s|^r+\frac{\varepsilon}{2}=h|t-s|^r+\varepsilon\end{aligned}$$

for sufficiently large n, and for every t and s in D. Since $\varepsilon>0$ is arbitrarily small, from (3.2) we have
$|X(t, w)-X(s, w)|\leq h|t-s|^r$ for every t and s in D. Hence $w\in C_h$, and so C_h is closed.

Now let us prove that C_h is sequentially compact with respect to the metric topology of (Ω, d_1). Let $\{w_n\}$ be a sequence in C_h. Then $\{X(\cdot, w_n)\}\subset B_h\subset A_h$. Hence $\{X(\cdot, w_n)\}$ has a convergent subsequence $\{X(\cdot, w_{n_i})\}$ since A_h is sequentially compact. Thus $\{X(\cdot, w_{n_i})\}$ is a Cauchy sequence *in* A_h. On the other hand, the inequality

$$d_1(w_{n_i}, w_{n_j})\leqq c\ d_2(X(\cdot, w_{n_i}), X(\cdot, w_{n_j}))$$

for some $c>0$, implies that $\{w_{n_i}\}$ is a Cauchy sequence in (Ω, d_1).

By the completeness of (Ω, d_1), there exists a point w in Ω such that $d_1(w_{n_i}, w)\to 0$ as $i\to\infty$. But $\{w_{n_i}\}$ is a sequence in C_h and C_h is closed. Hence w is in C_h. Thus every sequence in C_h has a convergent subsequence, and hence C_h is sequentially compact.

LEMMA 3.5. *Let $X(\cdot, \cdot)$ be a continuous Gaussian process on a complete probability space $(\Omega, \mathcal{F}, P)$ and $D=[a, b]\subset \mathbf{R}^1$, which satisfies the following conditions;*

(1) *mean $E[X(t, \cdot)]=0$ for every t in D.*

(2) *variance $V[X(t, \cdot)-X(s, \cdot)]\leqq B|t-s|^\beta$ for every t and s in D, where B and β are the given positive constant real numbers. Let $0<r<\frac{1}{2}\min(\beta, 1)$, where $\beta>0$. Then we have*

$$C_h \text{ is in } \mathcal{F}, \text{ and } P(C_h)>1-d\cdot h^{-\frac{4}{\beta-2r}} \text{ for all } h\geqq h_0,$$

where h_0 is some positive real number, and

$$d=2(b-a)^2\left[\frac{(1-2^{-r})^2(\beta-2r)e}{16B}\right]^{-\frac{2}{\beta-2r}}$$

Proof. Since we have shown that C_h is closed in the proof of Lemma 3.4, it is clear that C_h is in $\mathcal{F}$. Now let us prove the second part of the lemma. Subdivide the interval $[a, b]$ into the binary intervals $I(k\,;j)=[p^k(j-1),\ p^k(j)]$, where

$$p^k(j)=a+(b-a)\frac{j}{2^k} \text{ and } j=1, 2, \cdots, 2^k,\ \ k=0, 1, 2, \cdots.$$

Define $J_h(k, j)$ and S_h as follows;

$$J_h(k,j)=\left\{w\in\Omega : |X(p^k(j), w)-X(p^k(j-1), w)|>\frac{h(1-2^{-r})}{2}[p^k(j)-p^k(j-1)]^r\right\}$$

$$S_h=\bigcup_{k=0}^{\infty}\bigcup_{j=1}^{2^k} J_h(k,j),$$

To show that $\Omega-C_h=C_h^c\subset S_h$, assume that $w\in C_h^c$. Then there exist some points s and t in $D=[a, b]$ such that

(3.3) $$|X(t, w)-X(s, w)|>h|t-s|^r.$$

Since the set $\{p^k(j) : p^k(j)=a+(b-a)\frac{j}{2^k},\ j=0, 1, \cdots, 2^k,\ k=0, 1, \cdots\}$ is dense in $[a, b]$, we can choose k, j, k_0 and j_0 such that two points $p^k(j)$ and $p^{k_0}(j_0)$ are so close to t and s, respectively, that

(3.4) $$|X(p^k(j), w)-X(p^{k_0}(j_0), w)|>h|p^k(j)-p^{k_0}(j_0)|^r.$$

It is possible to induce (3.4) from (3.3) because of the continuity of $X(\cdot, w)$ for every w in Ω. Let n be a positive integer such that $n\geqq\max(k, k_0)$. Then the points $p^k(j)$ and $p^{k_0}(j_0)$ are contained in the set $\prod^0=\{p^n(0), \cdots, p^n(2^n)\}$. It follows from Proposition 2-4 that

$$|X(t, w)-X(s, w)|\leqq h\frac{(1-2^{-r})}{2}|t-s|^r$$

is false for some points t and s which are the endpoitnts of a primary binary interval. Hence $t=p^{k_1}(j_1-1)$ and $s=p^{k_1}(j_1)$ for some $k_1\leqq n$. Thus w is in $J_h(k_1, j_1)$ and hence w is in S_h. This shows that C_h^c is contained in S_h. But this means that

(3.5) $$P(C_h^c)\leqq P(S_h)\leqq\sum_{k=0}^{\infty}\sum_{j=1}^{2^k} P(J_h(k,j)).$$

On the other hand, by Proposition 2-5 we have

(3.6) $$P(J_h(k,j))<\frac{2}{h(1-2^{-r})}\left[\frac{2B}{\pi}\right]^{\frac{1}{2}}(b-a)^{\frac{\beta}{2}-r}\cdot\exp\left\{-\frac{h^2(1-2^{-r})^2}{8B|p^k(j)-p^k(j-1)|^{\beta-2r}}\right\}$$

Take a positive number h_0 such that

$$\frac{2}{h_0(1-2^{-r})}\left[\frac{2B}{\pi}\right]^{\frac{1}{2}}(b-a)^{\frac{\beta}{2}-r}\leqq1.$$

Then for all $h\geqq h_0$, we have from (3.6)

(3.7) $$P(J_h(k,j))<\exp\left\{-\frac{h^2(1-2^{-r})^2}{8B|p^k(j)-p^k(j-1)|^{\beta-2r}}\right\}$$

But, since $p^k(j)-p^k(j-1)=\frac{b-a}{2^k}$, we have from (3.7)

(3.8) $$P(J_h(k,j))<\exp\left\{-\frac{h^2(1-2^{-r})^2}{8B(b-a)^{\beta-2r}}\cdot2^{k(\beta-2r)}\right\}=\exp\{-H\cdot4^{k\lambda}\}$$

where $H=\frac{h^2(1-2^{-r})^2}{8B(b-a)^{\beta-2r}}$ and $\lambda=\frac{\beta-2\gamma}{2}$.

From (3.5) and (3.8), we obtain

(3.9) $$P(S_h)<\sum_{k=0}^{\infty}\sum_{j=1}^{2^k}\exp\{-H4^{k\lambda}\}$$
$$=\sum_{k=0}^{\infty}2^{-k}4^k\exp(-H4^{k\lambda})\ \text{for all } h\geqq h_0.$$

But we can easily show the fact that $f(u)=u\ \exp(-Hu^\lambda)$, $u\in R^1$, has the maximum value $(H\lambda e)^{-\frac{1}{\lambda}}$ at $u=(H\lambda)^{-\frac{1}{\lambda}}$. Hence we obtain from (3.9)

$$P(S_h)<\sum_{k=0}^{\infty}2^{-k}(H\lambda e)^{-\frac{1}{\lambda}}=2(H\lambda e)^{-\frac{1}{\lambda}}=d\cdot h^{-\frac{4}{\beta-2r}}$$

where $d=2(b-a)^2\left[\frac{(1-2^{-r})^2(\beta-2r)e}{16B}\right]^{-\frac{2}{\beta-2r}}$

Therefore $P(C_h{}^c)<d\cdot h^{-\frac{4}{\beta-2r}}$ for all $h\geqq h_0$, and hence $P(C_h)>1-d\cdot h^{-\frac{4}{\beta-2r}}$ for all $h\geqq h_0$.

COROLLARY. $p(\bigcup_{h=1}^{\infty}C_h)=1$, *where* C_h *is as in* Definition 3.3.

LEMMA 3.6. *Let* E *be any subset of* $\mathbf{R}^n$ *and let*
$K=\{w\in\Omega : (X(t_1, w),\cdots,X(t_n, w))\in E\}$ *be as in* Theorem 3.2.
Let G *be any open set in* Ω *which contains* K. *Let* $h>0$ *be arbitrarily given. Then there exists an open set* U *in* $\mathbf{R}^n$ *such that* $E\subset U$ *and*
$$\{w\in\Omega : w\in C_h \text{ and } (X(t_1, w),\cdots, X(t_n, w))\in U\}\subset G.$$

Proof. Case 1: Assume that E consists of a single point, say $E=\{P\}$, where $P=(p_1,\cdots,p_n)\in\mathbf{R}^n$. Suppose that the conclusion of the lemma is false. Then there exists a sequence $\{w_k, k=1,2,3,\cdots\}$ in C_h-G such that $\lim_{k\to\infty}X(t_j, w_k)=p_j$ where $j=1,2,\cdots n$. By Lemma 3.4, C_h is sequentially compact, and hence there exists a subsequence $\{w_{k_i}\}$ in C_h such that $w_{k_i}\to w$ as $i\to\infty$ for some w in C_h. Then $p_j=\lim_{i\to\infty}X(t_j, w_{k_i})=X(t_j, w)$, where $j=1,\cdots,n$. Thus $(X(t_1, w),\cdots,X(t_n, w))=(p_1,\cdots,p_n)=P\in E$ and hence w is in K. But since w_{k_i} is in G^c and G^c is closed, $\lim_{i\to\infty}w_{k_i}=w$ is in G^c. This is contrary to $K\subset G$. Therefore the lemma is established when E is a singleton set.

Case 2 (General case): Let E be an arbitrary set in $\mathbf{R}^n$. Using Case 1 we see that for each point P in E there exists an open set U_P in $\mathbf{R}^n$ which contains P such that
$$\{w\in\Omega : w\in C_h \text{ and } (X(t_1, w),\cdots,X(t_n, w))\in U_P\}\subset G.$$
Let $U=\bigcup_{P\in E}U_P$. Then U is an open set in $\mathbf{R}^n$ containing E such that
$$\{w\in\Omega : w\in C_h \text{ and } (X(t_1, w),\cdots, X(t_n, w))\in U\}$$
$$\subset\bigcup_{P\in E}\{w\in\Omega : w\in C_h \textit{ and } (X(t_1, w),\cdots,X(t_n, w))\in U_P\}\subset G.$$

REMARK. Given a Gaussian process $X(\cdot, \cdot)$ on a probability space $(S, \mathcal{S}, \nu)$ and $D=[a, b]\subset R^1$. Consider a random vector $X_t=(X(t_1, \cdot), \cdots, X(t_n, \cdot))$, where $t=(t_1, \cdots, t_n)$ and $a=t_0<t_1<\cdots<t_n=b$. Define a set function μ on the Borel σ-algebra $\mathcal{B}(\mathbf{R}^n)$ of $\mathbf{R}^n$ as follows;

$$\mu(B)=\nu(X_t^{-1}(B)) \text{ for every } B \text{ in } \mathcal{B}(\mathbf{R}^n).$$

Then μ is a probability measure on $\mathcal{B}(\mathbf{R}^n)$ and it is well known that μ is extended to a measure which is defined on the Lebesgue measurable sets in $\mathbf{R}^n$. From now on, we define that μ^* means the outer measure induced by μ and μ_* means the inner measure induced by μ.

LEMMA 3.7. *Let E be any subset of $\mathbf{R}^n$ and let $(\Omega, \mathcal{F}, P)$ and $J_t(E)=K$ be as in* Theorem 3-2. *If K is in $\mathcal{F}$, then $\mu^*(E)=P(K)$.*

Proof. Let $\tilde{E}$ be a Lebesgue measurable set such that $E\subset\tilde{E}$ and $\mu(\tilde{E})=\mu^*(E)$. Let $\tilde{K}=J_t(\tilde{E})$. Then $\tilde{K}$ is in $\mathcal{F}$ by Theorem 3.1, since $\tilde{E}$ is a Lebesgue measurable set, and $K\subset\tilde{K}$. This implies

$$(3.10) \qquad P(K)\leqq P(\tilde{K})=\mu(\tilde{E})=\mu^*(E).$$

Now it suffices to show that $\mu^*(E)\leqq P(K)$. Let $\varepsilon>0$ be arbitrarily given. We will show that $\mu^*(E)<P(K)+\varepsilon$. Since K is in $\mathcal{F}$, there exists a Borel set K_0 in Ω and a subset N_0 of N such that $P(N)=0$ and $K=K_0\cup N_0$. Let $G_0=K_0\cup N$. Then $K\subset G_0$ and G_0 is a Borel set in Ω. Hence we have

$$(3.11) \qquad P(K)=P(K_0\cup N_0)=P(K_0)=P(K_0\cup N)=P(G_0).$$

On the other hand, by Lemma 3.5, we can choose a positive number h so large that $P(C_h^c)<\frac{\varepsilon}{2}$. So we have

$$(3.12) \qquad P(G_0\cup C_h^c)\leqq P(G_0)+P(C_h^c)<P(K)+\frac{\varepsilon}{2}.$$

Since $G_0\cup C_h^c$ is a Borel set, it follows from Proposition 2.2 that there exists an open set G_1 containing $G_0\cup C_h^c$ such that

$$(3.13) \qquad P(G_1-(G_0\cup C_h^c))<\frac{\varepsilon}{2}.$$

Thus we have

$$(3.14) \qquad P(G_1)<P(G_0\cup C_h^c)+\frac{\varepsilon}{2}$$

by (3.13) and hence $P(G_1)<P(K)+\varepsilon$ with the help of (3.12).

Since $G_1\supset G_0\cup C_h^c\supset G_0\supset K$, it follows from Lemma 3.6 that there exists an open set U in $\mathbf{R}^n$ such that $E\subset U$ and $C_h\cap G\subset G_1$ where

$$G=\{w\in\Omega : (X(t_1, w), \cdots, (X(t_n, w))\in U\}.$$

Since U is open in $\mathbf{R}^n$, U is a Lebesgue measurable set. Thus G is in $\mathcal{F}$ by Theorem 3.1. From the facts that $G\cap C_h\subset G_1$ and $G\cap C_h^c\subset C_h^c\subset G_0\cup C_h^c\subset G_1$, we have also $G=(G\cap C_h)\cup(G\cap C_h^c)\subset G_1$. Now we come to a conclusion;

$$\mu^*(E) \leqq \mu(U) = P(G) \leqq P(G_1) < P(K) + \varepsilon.$$

This completes the proof of the lemma.

Proof of Theorem 3.2. $J_t(E) = K$ is in $\mathfrak{F}$ implies that $K^c = \{J_t(E)\}^c = J_t(E^c)$ is in $\mathfrak{F}$. By Lemma 3.7, we have $P(K^c) = \mu^*(E^c)$. Hence we have the following result;

(3.15) $$1 = P(\Omega) = P(K) + P(K^c) = \mu^*(E) + \mu^*(E^c).$$

On the other hand, by [5, 24 Corollary, p. 275] we have

(3.16) $$1 = \mu(\mathbf{R}^n) = \mu_*(E) + \mu^*(E^c).$$

From (3.15) and (3.16), we obtain $\mu^*(E) = \mu_*(E)$. By observing $\mu(N) = m(N)$ for every Borel null set N on $\mathbf{R}^n$, it can be shown that $\{E \subset \mathbf{R}^n : \mu^*(E) = \mu_*(E)\} = \{E \subset \mathbf{R}^n : E$ is Lebesgue measurable$\}$. Hence E is Lebesgue measurable. Thus the proof of Theorem 3.2 is complete.

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