J. Korean Math. Soc.
Vol. 19, No. 2, 1983

ON HADAMARD PRODUCTS FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

BY SHIGEYOSHI OWA

I. Introduction

Let $\mathcal{S}$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

which are analytic and univalent in the unit disk $\mathcal{U}=\{z : |z|<1\}$. A function $f(z)\in\mathcal{S}$ is said to be starlike of order α $(0\leqq\alpha<1)$, if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha$$

for $z\in\mathcal{U}$ and a function $f(z)\in\mathcal{S}$ is said to be convex of order α $(0\leqq\alpha<1)$, if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$$

for $z\in\mathcal{U}$. And we denote by $\mathcal{S}^*(\alpha)$ the class of all starlike functions of order α and $\mathcal{K}(\alpha)$ the class of all convex functions of order α.

Let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n\geqq 0).$$

We also denote by $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ the subclasses of $\mathcal{T}$ which are, respectively, starlike of order α $(0\leqq\alpha<1)$ in the unit disk $\mathcal{U}$ and convex of order $\alpha\,(0\leqq\alpha<1)$ in the unit disk $\mathcal{U}$.

For these classes, H. Silverman [6] showed the following lemmas.

LEMMA 1. *A function*

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n\geqq 0)$$

is in the class $\mathcal{T}^*(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n\leqq 1-\alpha.$$

The result is sharp.

Received June 26, 1982.

LEMMA 2. *A function*

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

is in the class $\mathcal{C}(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leqq 1-\alpha.$$

The result is sharp.

2. Properties of $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$

THEOREM 1. *Let* $0 \leqq \alpha_1 \leqq \alpha_2 < 1$. *Then we have* $\mathcal{T}^*(\alpha_1) \supset \mathcal{T}^*(\alpha_2)$.

Proof. Let a function

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

be in the class $\mathcal{T}^*(\alpha_2)$ and $\alpha_1=\alpha_2-\varepsilon$. Then, by using Lemma 1, we have

$$\sum_{n=2}^{\infty} (n-\alpha_2)a_n \leqq 1-\alpha_2$$

and

$$\sum_{n=2}^{\infty} a_n \leqq \frac{1-\alpha_2}{2-\alpha_2}.$$

Hence we get

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\alpha_1)a_n &= \sum_{n=2}^{\infty}(n-\alpha_2+\varepsilon)a_n = \sum_{n=2}^{\infty}(n-\alpha_2)a_n + \varepsilon\sum_{n=2}^{\infty}a_n \\ &\leqq 1-\alpha_2+\frac{\varepsilon(1-\alpha_2)}{2-\alpha_2} \leqq 1-\alpha_2+\varepsilon = 1-\alpha_1.\end{aligned}$$

This shows that $f(z)$ is in the class $\mathcal{T}^*(\alpha_1)$ by means of Lemma 1.

THEOREM 2. *Let* $0 \leqq \alpha_1 \leqq \alpha_2 < 1$. *Then we have*

$$\mathcal{C}(\alpha_1) \supset \mathcal{C}(\alpha_2).$$

The proof of Theorem 2 is obtained by using the same technique as in the proof of Theorem 1 with the aid of Lemma 2.

3. Hadamard products

Let $f*g(z)$ denote the Hadamard product of two functions

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

and

$$g(z)=z-\sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geqq 0),$$

that is,

$$f*g(z)=z-\sum_{n=2}^{\infty}a_nb_nz^n.$$

THEOREM 3. *Let the functions*

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{n,i}z^n \qquad (a_{n,i}\geqq 0)$$

be in the same class $\mathcal{T}^*(\alpha)$ *for every* $i=1,2,\cdots, m$. *Then the Hadamard product* $f_1*f_2*\cdots*f_m(z)$ *belongs to the class* $\mathcal{T}^*(1-(1-\alpha)^m/(2-\alpha)^{m-1})$.

Proof. Since $f_i(z)\in\mathcal{T}^*(\alpha)$ for every $i=1,2,\cdots m$, by virtue of Lemma 1, we have

$$\sum_{n=2}^{\infty}(n-\alpha)a_{n,1}\leqq 1-\alpha$$

and

$$a_{n,i}\leqq\frac{1-\alpha}{2-\alpha}$$

for $n\geqq 2$ and $i=2,3,\cdots,m$. Therefore we get

$$\begin{aligned}\sum_{n=2}^{\infty}\left\{n-\left\{1-\frac{(1-\alpha)^m}{(2-\alpha)^{m-1}}\right\}\right\}\prod_{i=1}^{m}a_{n,i}&\\ \leqq\sum_{n=2}^{\infty}(n-\alpha)\prod_{i=1}^{m}a_{n,i}\leqq\frac{(1-\alpha)^m}{(2-\alpha)^{m-1}}&\\ =1-\left\{1-\frac{(1-\alpha)^m}{(2-\alpha)^{m-1}}\right\}.&\end{aligned}$$

Further

$$0<1-\frac{(1-\alpha)^m}{(2-\alpha)^{m-1}}<1$$

for $0\leqq\alpha<1$. This gives that the Hadamard product $f_1*f_2*\cdots*f_m(z)$ is in the class $\mathcal{T}^*(1-(1-\alpha)^m/(2-\alpha)^{m-1})$.

COROLLARY 1. *Under the hypotheses of Theorem* 3, *we have* $f_1*f_2*\cdots*f_m(z)\in\mathcal{T}^*(\alpha)$.

Proof. Since

$$\alpha<1-\frac{(1-\alpha)^m}{(2-\alpha)^{m-1}}$$

for $0\leqq\alpha<1$, it is clear that $f_1*f_2*\cdots*f_m(z)\in\mathcal{T}^*(\alpha)$ with the aid of Theorem 1.

THEOREM 4. *Let the functions*

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{n,i}z^n \qquad (a_{n,i}\geqq 0)$$

be in the same class $\mathcal{C}(\alpha)$ *for every* $i=1,2,\cdots,m$. *Then the Hadamard product* $f_1*f_2*\cdots*f_m(z)$ *belongs to the class* $\mathcal{C}(1-(1-\alpha)^m/2^{m-1}(2-\alpha)^{m-1})$.

Proof. Since $f_i(z)\in\mathcal{C}(\alpha)$ for every $i=1,2,\cdots,m$, in conjunction with Lemma 2,

$$\sum_{n=2}^{\infty} n(n-\alpha)a_{n,1}\leqq 1-\alpha$$

and

$$a_{n,i}\leqq\frac{1-\alpha}{2(2-\alpha)}$$

for $n\geqq 2$ and $i=1,2,\cdots,m$. Therefore we obtain

$$\sum_{n=2}^{\infty} n\left\{n-\left\{1-\frac{(1-\alpha)^m}{2^{m-1}(2-\alpha)^{m-1}}\right\}\right\}\prod_{i=1}^{m}a_{n,i}$$
$$\leqq\sum_{n=2}^{\infty}n(n-\alpha)\prod_{i=1}^{m}a_{n,i}\leqq\frac{(1-\alpha)^m}{2^{m-1}(2-\alpha)^{m-1}}$$
$$=1-\left\{1-\frac{(1-\alpha)^m}{2^{m-1}(2-\alpha)^{m-1}}\right\}.$$

Again we have

$$0<1-\frac{(1-\alpha)^m}{2^{m-1}(2-\alpha)^{m-1}}<1$$

for $0\leqq\alpha<1$. This shows that $f_1*f_2*\cdots*f_m(z)$ is in the class $\mathcal{C}(1-(1-\alpha)^m/2^{m-1}(2-\alpha)^{m-1})$.

COROLLARY 2. *Under the hypotheses of Theorem 4, we have* $f_1*f_2*\cdots*f_m(z)\in\mathcal{C}(\alpha)$.

The proof of Corollary 2 is evident by using Theorem 2.

THEOREM 5. *Let the functions*

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{n,i}z^n \qquad (a_{n,i}\geqq 0)$$

be in the same class $\mathcal{T}^*(\alpha)$ *for every* $i=1,2,\cdots m$. *Further let the functions*

$$g_j(z)=z-\sum_{n=2}^{\infty}b_{n,j}z^n \qquad (b_{n,j}\geqq 0)$$

be in the same class $\mathcal{C}(\alpha)$ *for every* $j=1,2,\cdots,p$. *Then the Hadamard product* $f_1*f_2*\cdots*f_m*g_1*g_2*\cdots*g_p(z)$ *belongs to the class* $\mathcal{T}^*(1-(1-\alpha)^{m+p}/2^p(2-\alpha)^{m+p-1})$.

Proof. Since $f_i(z)\in\mathcal{T}^*(\alpha)$ for every $i=1,2,\cdots,m$, and $g_j(z)\in\mathcal{C}(\alpha)$ for every $j=1,2,\cdots,p$, in virtue of Lemma 1 and Lemma 2, we have

$$\sum_{n=2}^{\infty}(n-\alpha)a_{n,1}\leqq 1-\alpha, \qquad a_{n,i}\leqq\frac{1-\alpha}{2-\alpha}$$

for $n\geqq 2$ and $i=2,3,\cdots,m$, and

$$b_{n,j}\leqq\frac{1-\alpha}{2(2-\alpha)}$$

for $n \geqq 2$ and $j=1, 2, \cdots, p$. Hence we obtain

$$\sum_{n=2}^{\infty}\left\{n-\left\{1-\frac{(1-\alpha)^{m+p}}{2^{p}(2-\alpha)^{m+p-1}}\right\}\right\}\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{p}b_{n,j}$$

$$\leqq \sum_{n=2}^{\infty}(n-\alpha)\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{p}b_{n,j} \leqq \frac{(1-\alpha)^{m+p}}{2^{p}(2-\alpha)^{m+p-1}}$$

$$=1-\left\{1-\frac{(1-\alpha)^{m+p}}{2^{p}(2-\alpha)^{m+p-1}}\right\}$$

and

$$0<1-\frac{(1-\alpha)^{m+p}}{2^{p}(2-\alpha)^{m+p-1}}<1$$

for $0 \leqq \alpha < 1$, $m \in \mathcal{N}$ and $p \in \mathcal{N}$. This completes the proof of the theorem.

COROLLARY 3. *Under the hypotheses of Theorem* 5, *we have* $f_1*f_2*\cdots*f_m*g_1*g_2*\cdots*g_p(z) \in \mathcal{T}^*(\alpha)$.

THEOREM 6. *Let the functions*

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{n,i}z^n \qquad (a_{n,i}\geqq 0)$$

be in the same class $\mathcal{T}^*(\alpha)$ *for every* $i=1, 2, \cdots, m$. *Further let the functions*

$$g_j(z)=z-\sum_{n=2}^{\infty}b_{n,j}z^n \qquad (b_{n,j}\geqq 0)$$

be in the same class $\mathcal{C}(\alpha)$ *for every* $j=1, 2, \cdots, p$. *Then the Hadamard product* $f_1*f_2*\cdots*f_m*g_1*g_2*\cdots*g_p(z)$ *belongs to the class* $\mathcal{C}(1-(1-\alpha)^{m+p}/2^{p-1}(2-\alpha)^{m+p-1})$.

Proof. Since $f_i(z) \in \mathcal{T}^*(\alpha)$ for every $i=1, 2, \cdots, m$, and $g_j(z) \in \mathcal{C}(\alpha)$ for every $j=1, 2, \cdots, p$, by using Lemma 1 and Lemma 2,

$$a_{n,i} \leqq \frac{1-\alpha}{2-\alpha}$$

for $n \geqq 2$ and $i=1, 2, \cdots, m$,

$$\sum_{n=2}^{\infty}n(n-\alpha)b_{n,1} \leqq 1-\alpha$$

and

$$b_{n,j} \leqq \frac{1-\alpha}{2(2-\alpha)}$$

for $n \geqq 2$ and $j=2, 3, \cdots, p$. Consequently we have

$$\sum_{n=2}^{\infty}n\left\{n-\left\{1-\frac{(1-\alpha)^{m+p}}{2^{p-1}(2-\alpha)^{m+p-1}}\right\}\right\}\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{p}b_{n,j}$$

$$\leqq \sum_{n=2}^{\infty}n(n-\alpha)\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{p}b_{n,j} \leqq \frac{(1-\alpha)^{m+p}}{2^{p-1}(2-\alpha)^{m+p-1}}$$

$$=1-\left\{1-\frac{(1-\alpha)^{m+p}}{2^{p-1}(2-\alpha)^{m+p-1}}\right\}.$$

Further

$$0<1-\frac{(1-\alpha)^{m+p}}{2^{p-1}(2-\alpha)^{m+p-1}}<1$$

for $0\leqq\alpha<1$, $m\in\mathscr{N}$ and $p\in\mathscr{N}$. This shows that the Hadamard product $f_1*f_2*\cdots*f_m*g_1*g_2*\cdots*g_p(z)$ is in the class $\mathscr{C}(1-(1-\alpha)^{m+p}/2^{p-1}(2-\alpha)^{m+p-1})$.

COROLLARY 4. *Under the hypotheses of Theorem 6, we have* $f_1*f_2*\cdots*f_m*g_1*g_2*\cdots g_p(z)\in\mathscr{C}(\alpha)$.

THEOREM 7. *Let* $0\leqq\alpha\leqq 1/2$. *Further let the functions*

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{n,i}z^n \qquad (a_{n,i}\geqq 0)$$

be in the same class $\mathscr{T}^*(\alpha)$ *for every* $i=1,2,\cdots,m$. *Then the Hadamard product* $f_1*f_2*\cdots*f_m(z)$ *belongs to the class* $\mathscr{C}(1-(1-\alpha)^m/(2-\alpha)^{m-2})$.

Proof. Let $f_i(z)\in\mathscr{T}^*(\alpha)$ for every $i=1,2,\cdots,m$, by Lemma 1, we get

$$\sum_{n=2}^{\infty}(n-\alpha)a_{n,1}\leqq 1-\alpha, \quad (n-\alpha)a_{n,2}\leqq 1-\alpha$$

for $n\geqq 2$ and

$$a_{n,i}\leqq\frac{1-\alpha}{2-\alpha}$$

for $n\geqq 2$ and $i=3,4,\cdots,m$. Consequently we obtain

$$\begin{aligned}\sum_{n=2}^{\infty}n\left\{n-\left\{1-\frac{(1-\alpha)^m}{(2-\alpha)^{m-2}}\right\}\right\}\prod_{i=1}^{m}a_{n,i}\\ \leqq\sum_{n=2}^{\infty}(n-\alpha)^2\prod_{i=1}^{m}a_{n,i}\leqq\frac{(1-\alpha)^m}{(2-\alpha)^{m-2}}\\ =1-\left\{1-\frac{(1-\alpha)^m}{(2-\alpha)^{m-2}}\right\}\end{aligned}$$

and

$$0<1-\frac{(1-\alpha)^m}{(2-\alpha)^{m-2}}<1$$

for $0\leqq\alpha\leqq 1/2$ and $m\geqq 2$. Hence we have the theorem with the aid of Lemma 2.

COROLLARY 5. *Under the hypotheses of* Theorem 7, *we have* $f_1*f_2*\cdots*f_m(z)\in\mathscr{C}(\alpha)$.

THEOREM 8. *Let the functions*

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{n,i}z^n \qquad (a_{n,i}\geqq 0)$$

be in the same class $\mathscr{C}(\alpha)$ *for every* $i=1,2,\cdots,m$. *Then the Hadamard product* $f_1*f_2*\cdots*f_m(z)$ *belongs to the class* $\mathscr{T}^*(1-(1-\alpha)^m/2^m(2-\alpha)^{m-1})$.

Proof. Since $f_i(z) \in \mathcal{C}(\alpha)$ for every $i=1, 2, \cdots, m,$ in virtue of Lemma 2, we get

$$\sum_{n=2}^{\infty}(n-\alpha)a_{n,1} \leqq \frac{1-\alpha}{2}$$

and

$$a_{n,i} \leqq \frac{1-\alpha}{2(2-\alpha)}$$

for $n \geqq 2$ and $i=2, 3, \cdots, m.$ Therefore

$$\sum_{n=2}^{\infty}\left\{n-\left\{1-\frac{(1-\alpha)^m}{2^m(2-\alpha)^{m-1}}\right\}\right\}\prod_{i=1}^{m} a_{n,i}$$
$$\leqq \sum_{n=2}^{\infty}(n-\alpha)\prod_{i=1}^{m} a_{n,i} \leqq \frac{(1-\alpha)^m}{2^m(2-\alpha)^{m-1}}$$
$$=1-\left\{1-\frac{(1-\alpha)^m}{2^m(2-\alpha)^{m-1}}\right\}$$

and

$$0<1-\frac{(1-\alpha)^m}{2^m(2-\alpha)^{m-1}}<1$$

for $0 \leqq \alpha < 1$ and $m \in \mathcal{N}.$ This gives that $f_1 * f_2 * \cdots * f_m(z)$ is in the class $\mathcal{T}^*(1-(1-\alpha)^m/2^m(2-\alpha)^{m-1})$ with Lemma 1.

COROLLARY 6. *Under the hypotheses of* Theorem 8, $f_1 * f_2 * \cdots * f_m(z) \in \mathcal{T}^*(\alpha).$

REMARK. For Hadamard products of other classes of analytic and univalent functions in the unit disk $\mathcal{U}$, S. Owa [3], [4] showed some results.

4. The class $\mathcal{C}_1(\alpha)$

Let $\mathcal{K}_1(\alpha)$ be the class of function

$$F(z)=\frac{1}{2}\{f(z)+zf'(z)\},$$

where $f(z) \in \mathcal{K}(\alpha).$

The class $\mathcal{K}_1(0)$ was studied by B. N. Rahmanov [5], A. E. Livingston [2], R. M. Goel [1] and V. Singh and R. Singh [7].

In this place, let $\mathcal{C}_1(\alpha)$ be the class of function

$$G(z)=\frac{1}{2}\{f(z)+zf'(z)\},$$

where $f(z) \in \mathcal{C}(\alpha).$

THEOREM 9. *Let*

$$G(z)=\frac{1}{2}\{f(z)+zf'z)\}$$

be in the class $\mathcal{C}_1(\alpha)$. *Then* $G(z)\in\mathcal{T}^*(\alpha)$.

Proof. Let

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \qquad (a_n\geqq 0),$$

then the function $G(z)$ has the expansion

$$G(z)=z-\sum_{n=2}^{\infty}\left(\frac{n+1}{2}\right)a_n z^n.$$

Since $f(z)\in\mathcal{C}(\alpha)$, in conjunction with Lemma 2, we have

$$\sum_{n=2}^{\infty}(n-\alpha)\left(\frac{n+1}{2}\right)a_n\leqq\sum_{n=2}^{\infty}n(n-\alpha)a_n\leqq 1-\alpha$$

which shows that $G(z)$ belongs to the class $\mathcal{T}^*(\alpha)$.

References

1. R. M. Goel, *On a class of functions schlicht in the unit circle*, Rev. Math. Hisp-Amer. **31** (1971), 20–33.
2. A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **17** (1966), 352–357.
3. S. Owa, *A remark on the Hadamard products of starlike functions II*, Math. Japonica, **27** (1982), 747–752.
4. S. Owa, *On the Hadamard products of univalent functions*, Tamkang J. Math. **14** (1983). (to appear).
5. B. N. Rahmanov, *On the theory of univalent functions*, Dokl. Akad. Nauk. USSR **78** (1951), 209–211.
6. H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.
7. V. Singh and R. Singh, *On a class of functions schlicht in the unit disc*, Indian J. Pure Appl. Math. **7** (1975), 116–120.

Department of Mathematics
Kinki University
Osaka, Japan