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Metric Antiprojections and Characterization of ε-Farthest Points

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In the first section of this note we have studied the metric antiprojections for nearly compact sets and in the second section we have introduced the notion of ε-farthest points and have given a characterization of such points.

1. Metric Antiprojections for Nearly Compact Sets.

Let X be a normed linear space and K a non-empty bounded subset of X. The map

$$Q : X \longrightarrow K$$

where

$$Q(x) = \{y \in K : \|x-y\| = \sup_{z \in K} \|x-z\|\} \in F_K(x)$$

is called the *metric antiprojection* or the *farthest point map* associated with the set K. Every element $y \in Q(x)$ is called a *farthest point* of K for x. K is called *remotal* [7] if $Q(x) \neq \phi$ for each $x \in X$ and *uniquely remotal* [7] if $Q(x)$ consists of exactly one point for each $x \in X$. A sequence $\langle g_n \rangle$ in K is called a *maximizing sequence* for x if

$$\lim_{n \to \infty} \|x - g_n\| = \sup_{z \in K} \|x - z\|.$$

If for each $x \in X$ every maximizing sequence for x has a subsequence converging to an element of K then K is called *nearly compact*[7] (or *M-compact*[8] or ^compact[2]). Clearly every compact set is nearly compact but not conversely (cf. [7]).

A subset A of a metric space X is called *residual* in X if $X \setminus A$ is a set of first Baire category in X. According to the classical Baire's theorem, any residual subset of a complete metric space is everywhere dense in the space.

Edelstein [4] proved that if X is a uniformly convex Banach space and K is a closed bounded subset of X then except on a set of first Baire category, the points in X have farthest points in K. The theorem was generalized by Asplund [1] to reflexive locally uniformly convex spaces and by Panda and Kapoor [9] to reflexive CLUR-spaces. Ka-Sing-Lau [6] proved that the result is true for any weakly compact subset of a Banach space. Using this result of Ka-Sing-Lau, Zhivkov [10] proved that if K is a weakly compact subset of a strictly convex smooth Banach space X then except on a set of first Baire category the points of X have unique farthest points in K and it was remarked that Asplund's result can be presented in the following stronger form: Antiprojections generated by any closed and bounded subset of a reflexive locally uniformly convex Banach space is single-valued except on a set of first Baire category.

We give below the following variant of this result:

Theorem: *Let K be a nearly compact subset of a strictly convex Banach space X and let $Q: X \to K$ be the antiprojection. Then Q is single valued and continuous on a residual part of the space X i.e. on a set dense in the space and consequently Q is uniquely remotal with respect to a dense subset of the space.*

The following two lemmas will be used in the proof.

Lemma 1 [10]: *Let $F: X \to Y$ be a multivalued mapping from a topological space X into a metric space Y. Suppose the following condition (α) is satisfied:*

[(α) *If $F(x) \neq \phi$ for $x \in X$ then for every open $V \ni x$ a point $z \in V$ exists such that $F(z) \neq \phi$ and F is both single-valued and upper semi-continuous at z.*] *Then the set $ES_F = \{x \in X$: the set $F(x)$ is either empty or singleton$\}$ is residual in X.*

Lemma 2 [2]: *If K is a nearly compact subset of a Banach space X then $Q: X \to K$ is upper semi-continuous and has nonempty images.*

Proof of Theorem: Let $x_0 \in X$. Since K is nearly compact, $Qx_0 \neq \phi$ (Lemma 2). Let $y_0 \in Qx_0$. Since X is strictly convex, at every point x from the set

$$\{x_t:\ x_t = x_0 + t(x_0 - y_0) : t > 0\}$$

the antiprojection Q is single-valued [10]. Moreover, $\lim x_t = x_0$ when $t \to 0$ and this means that we can find points arbitrary close to x_0 at which Q is both single-valued and upper semicontinuous (Lemma 2). Then by Lemma 1, the set $S_Q = \{x \in X$: Qx has exactly one element$\}$ is residual in X i.e. there exist a subset D dense in X such that Q is single-valued on D and consequently continous on D (for single-valued mappings upper semi-continuity is same as continuity).

Remark: For compact sets the validity of the above result was remarked by Zhivkov [10].

2. Characterization of ε-Farthest Points.

In the theory of nearest points R.C. Buck [3] introduced the notion of 'elements of ε-approximation' (good approximation) and gave a characterization of elements of ε-approximation. In this section we introduce an analogous notion 'elements of ε-farthest points' and give a characterization of such elements. In the particular case, when $\varepsilon = 0$, we get a characterization of farthest points given in [5]-Theorem 3.1.

Let K be a bounded subset of a normed linear space X, $x \in X$ and $\varepsilon > 0$. An element $k_0 \in K$ is said to be *ε-farthest point of x* (by means of elements of K) if

$$\|x - k_0\| \geqslant \sup\ \{\|x - y\| : y \in K\} - \varepsilon$$

We shall denote by $F_K(x, \varepsilon)$ the set of all elements of ε-farthest points of x. In particular, for $\varepsilon = 0$. We find again the elements of farthest points of x and respectively the sets $F_K(x)$. One of the advantages of considering the sets $F_K(x, \varepsilon)$ with $\varepsilon > 0$, instead of the sets $F_K(x)$ is that the sets $F_K(x, \varepsilon)$ are always non-empty for $\varepsilon > 0$ and K bounded.

The following theorem gives a characterization of ε-farthest points.

Theorem: *Let X be a normed linear space, K a bounded subset of X and $x \in X$. For an element $k_0 \in K$ and $\varepsilon > 0$ the following statements are equivalent:*

1^0. $k_0 \in F_K(x, \varepsilon)$

2^0. *There exists $f_0 \in X^*$ such that*

$$\|f_0\| = 1 \qquad (2.1)$$

$$f_0(x-k_0) \geqslant \sup_{y \in K} \|x-y\| - \varepsilon \tag{2.2}$$

3^0. *There exist* $f_0 \in X^*$ *satisfying* (2.1) *and*

$$|f_0 \ (x-k_0)| \geqslant \sup_{y \in K} \|x-y\| - \varepsilon \tag{2.3}$$

Proof: $1^0 \Longrightarrow 2^0$. By a Corollary to Hahn Banach theorem there exists $f_0 \in X^*$ such that

$$\|f_0\| = 1$$

and $f_0 \ (x-k_0) = \|x-k_0\| \geqslant \sup_{y \in K} \|x-y\| - \varepsilon$

$2^0 \Longrightarrow 3^0$ is obvious.

$3^0 \Longrightarrow 1^0$ consider

$$\|x-k_0\| \geqslant |f_0(x-k_0)| \geqslant \sup_{y \in K} \|x-y\| - \varepsilon$$

Thus $k_0 \in F_K(x, \varepsilon)$.

Remark: In the particular case when $\varepsilon = 0$, above theorem reduces to Theorem 3.1 [5] on the characterization of elements of farthest points.

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