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# Extensions of the Sum of Integers Formula

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Beginning with the sum of $n$ ones, which we denote as $S_n^{(1)}=1+1+\dots+1=\frac{n}{1}$, and then considering the formula $S_n^{(2)}=1+2+\dots+n=\frac{n(n+1)}{2}$, we typically investigate the sum of squares, then the sum of cubes, etc. The shortcoming of this process is that the formulas found or proved are actually quite different in form. It would seem that an alternative natural path to take would be to consider sums of $n$ terms which respectively equal $\frac{n(n+1)(n+2)}{3}$, $\frac{n(n+1)(n+2)(n+3)}{4}$, ... $\frac{1}{k}\prod_{i=0}^{k-1}(n+i)$, where $\prod_{i=0}^{k-1}(n+i)$ denotes the product $(n+0)(n+1)\dots(n+k-1)$. The article realizes such a development.

The process of going from one sum to the next is elementary once we observe that the second sum may be considered as an extension of the first. We write

$$S_n^{(2)}=1+2+\dots+n=1\cdot1+1\cdot2+\dots+1\cdot n,$$

and hence a possible next sum may be

$$S_n^{(3)}=1\cdot1\cdot2+1\cdot2\cdot3+\dots+1\cdot n\cdot(n+1).$$

To evaluate this sum, we use the earlier given formulas found in the hierarchy

$$S_n^{(3)}=2\left(\frac{1\cdot2}{2}+\frac{2\cdot3}{2}+\frac{3\cdot4}{2}+\dots+\frac{n(n+1)}{2}\right)$$

$$=2(1+(1+2)+(1+2+3)+\dots+(1+2+\dots+n)),$$

using $S_m^{(2)}=\frac{m(m+1)}{2}$,

$$=2(n(1)+(n-1)2+(n-2)3+\dots+(n-(n-1))n),$$

collecting, respectively, the 1's, 2's, ...,$n$'s, of the previous sum,

$$=2(n(1+2+\dots+n)-(1\cdot2+2\cdot3+\dots+(n-1)n))$$

$$=2\left(n\frac{n(n+1)}{2}-(S_n^{(3)}-n(n+1))\right).$$

Therefore, $S_n^{(3)}=\frac{n(n+1)(n+2)}{3}$.

Continuing, let $S_n^{(4)}=1\cdot1\cdot2\cdot3+1\cdot2\cdot3\cdot4+1\cdot3\cdot4\cdot5+\dots+1\cdot n\cdot(n+1)\cdot(n+2)$, and expressing the addends in terms of "$S_n$" notation, we have

$$S_n^{(4)}=3(S_1^{(3)}+S_2^{(3)}+S_3^{(3)}+\dots+S_n^{(3)})$$

$$=3((1\cdot1\cdot2)+(1\cdot1\cdot2+1\cdot2\cdot3)+(1\cdot1\cdot2+1\cdot2\cdot3+1\cdot3\cdot4)+\dots+(1\cdot1\cdot2+1\cdot2\cdot3+1\cdot3\cdot4+\dots+1\cdot n\cdot(n+1)))$$

$$=3(n(1\cdot2)+(n-1)(2\cdot3)+(n-2)(3\cdot4)+\dots+(n-(n-1))(n(n+1))),$$

by collecting like terms,

$$=3(n(1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1))-(1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+(n-1)n(n+1)))$$

$$=3(n\cdot S_n^{(3)}-(S_n^{(4)}-n(n+1)(n+2))).$$

Thus, $S_n^{(4)}=\frac{1}{4}n(n+1)(n+2)(n+3)$.

Now, if ever, we are prepared for the "inductive leap" which we glorify to theorem status.

**Theorem.** Let $n$ and $m\geq 2$ be positive integers. If we define $S_n^{(m)}=\sum_{i=1}^{n} 1(i)(i+1)\ldots\ldots(i+m-2)$, then $S_n^{(m)}=\frac{1}{m}\prod_{i=0}^{m-1}(n-i)$.

**Proof.** We proceed by induction on $m$, noting the theorem has already been shown valid for $m=2, 3$, and 4. Assume as the induction hypothesis that for any positive integer $n$, $S_n^{(k)}=\frac{1}{k}\prod_{i=0}^{k-1}(n+i)$ where $k\geq 2$ is a fixed but arbitrarily chosen positive integer. To show: $S_n^{(k+1)}=\frac{1}{k+1}\prod_{i=0}^{k}(n+i)$. From the given definition,

$$S_n^{(k+1)}=\sum_{i=1}^{n} 1(i)(i+1)\ldots(i+k-1)$$

$$=1\cdot 1\cdot 2\cdot 3\ldots(k)+1\cdot 2\cdot 3\cdot(k+1)+1\cdot 3\cdot 4\ldots(k+2)+\ldots+1\cdot n(n+1)(n+2)\ldots(n+k-1)$$

$$=k(S_1^{(k)}+S_2^{(k)}+S_3^{(k)}+\ldots+S_n^{(k)}),\text{ by induction hypothesis}$$

$$=k\left(\sum_{i=1}^{1} 1(i)(i+1)\ldots(i+k-2)+\sum_{i=1}^{2} 1(i)(i+1)\ldots(i+k-2)+\sum_{i=1}^{3}1(i)(i+1)\ldots(i+k-2)+\ldots+\sum_{i=1}^{n} 1(i)(i+1)\ldots(i+k-2)\right)$$

$$=k((n(1\cdot 2\ldots(k-1))+(n-1)(2\cdot 3\ldots k)+(n-2)(3\cdot 4\ldots(k+1))+\ldots+(n-(n-1))(n(n+1)\ldots(n+k-2))),\text{ by grouping like terms}$$

$$=k(nS_n^{(k)}-(S_n^{(k+1)}-n(n+1)\ldots(n+k-1)).$$

Hence,

$$(k+1)S_n^{(k+1)}=k\left(n(\frac{1}{k}\prod_{i=0}^{k-1}(n+i))+k\prod_{i=0}^{k-1}(n+i)\right)\text{using induction hypothesis again,}$$

$$=\left(\prod_{i=0}^{k-1}(n+i)\right)(n+k),$$

and so,

$$S_n^{(k+1)}=\frac{1}{k+1}\prod_{i=0}^{k}(n+i)\text{ completing the proof.}$$