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A Note on the Gelfand Representation Theory

By Shin Po Kang
Graduate School of Pusan National University, Pusan, Korea

1. Introduction

Let R be a commutative Banach algebra with unit e such that $\|e\|=1$, $\mathfrak{M}$ be the compact topological space of its maximal ideals. For a given x in R, $\hat{x}:\mathfrak{M}\to C$; $\hat{x}(\phi)=\phi(x)$. We introduce a topology in $\mathfrak{M}$ with the aid of the function $\hat{x}$. Then the resulting topology is Hausdorff space. But since each nonzero multiplicative linear functional ϕ is continuous on R with $\|\phi\|=1$, $\mathfrak{M}$ is a subset of the unit ball Σ in the conjugate space R^*. If we consider the weak* topology over R^* and then consider the topology it induces on $\mathfrak{M}$, the topology defined on $\mathfrak{M}$ is the trace on $\mathfrak{M}$ of the weak* topology defined in R^*. Thus, each $\hat{x}$ is continuous and, since $\mathfrak{M}$ is compact, $\hat{x}$ is bounded on $\mathfrak{M}$, that is, $\hat{x}\in C(\mathfrak{M})$. And in fact, $\mathfrak{M}$ has the weakest togology for which the mapping $\hat{x}$ are continuous. The correspondence $x\to\hat{x}$ is called *the Gelfand representation of R*. The Gelfand representation is a continuous homomorphism of R onto a subalgebra $\hat{R}$ cf $C(\mathfrak{M})$, since $\|\hat{x}\|_\infty=r_\sigma(x)\leqq\|x\|$.

This is an expository paper to establish the fundamental Gelfand representation theorem which characterizes those commutative Banach algebra that are isomorphic to an algebra of continuous functions on a compact Hausdorff space.

2. Preliminary Results

Let R be a Banach algebra. Let Φ be the transformation of R into $\mathfrak{A}(R)$ defined by $\Phi(a)=T_a$ where $T_a(x)=ax$ and $\mathfrak{A}(R)$ is the totality of bounded linear transform from R into R. Then Φ is an algebraic and topological isomorphism. If the complex Banach algebra R is a field, then R is isomorphic to the field of complex numbers. Let R be a Banach algebra such that $\|fg\|=\|f\|\|g\|$ for each pair $f,g\in R$. Then R is isomorphic to the field of complex numbers.

It is a well known fact that R/I is a field if and only if I is a maximal ideal. Thus if I is a closed ideal in R, if $I'=\{I+f\}$ is an element of $R'=R/I$ define $\|f'\|=\inf_{x\in I}\|x+f\|$, then $\|f'\|$ is a norm in R', with respect to this norm R' is complete and satisfying $\|f'g'\|\leqq\|f'\|\|g'\|$; thus R' is a Banach algebra.

Lemma 2.1. *Let I be a maximal ideal in R. The quotient ring $R'=R/I$ is isomorphic to the field of complex numbers.*

If $f\in R$, there exists one and only one complex number λ—we write $\lambda=f(I)$—such that $\{I+f\}=\{I+\lambda e\}$, equivalently expressed, $f\equiv\lambda(\text{mod } I)$.

Lemma 2.2. *Let F be a multiplicative linear functional of R onto the complex numbers and let I* *be the kernel of F, that is, $I=\{f:Ff=0\}$. Then I is a maximal ideal. Conversely, let I be a maxim ideal and Let F be the mapping $Ff=f'$ where $f'=\{I+f\}$ of R onto R/I. Then F is multiplicati linear functional of R onto the complex numbers and the kernel of F is I.*

The Lemma states that the notions of multiplicative linear functional and maximal ideal a equivalent. Cutting linquistic corners we shall frequently consider them identical.

Definition 2.3. A *carrier space of R* is the set of all nonzero multiplicative linear functionals o R, endowed with the topology of pointwise convergence on R.

For $x\in R$, the *Gelfand transformation of x* is the function $\hat{x}$ defined on $\mathfrak{M}$ by $\hat{x}(\phi)=\phi(x)$ ($\phi\in$ $\mathfrak{M}$). Let $\hat{R}$ be the set of all $\hat{x}$, for $x\in R$. The *Gelfand topology* on $\mathfrak{M}$ is the weak topology induce by $\hat{R}$, that is, the weakest topology that makes every $\hat{x}$ continuous. Then obviously $\hat{R}\subset C(\Delta)$, th algebra of all complex continuous functions on $\mathfrak{M}$.

Lemma 2.4. *Let R be a commutative Banach algebra with a unit e. Then the carrier space $\mathfrak{M}$ o R is a compact Hausdorff space.*

The correspondence Ω; $x\to\hat{x}$ is called the *Gelfand representation of R*. The mapping is line multiplicative function. Thus Ω is a homomorphism of R onto a subalgebra $\hat{R}$ of $C(\mathfrak{M})$. Denotir the norm in $C(\mathfrak{M})$ by $\|\cdot\|_\infty$. We have $\|\hat{x}\|_\infty=\sup_{\phi\in\mathfrak{M}}|\hat{x}(\phi)|=\sup_{\phi\in\mathfrak{M}}|\phi(x)|$. since $\phi(x)\in\sigma(x)$ for each ϕ $\|\hat{x}\|_\infty\leqq r_\sigma(x)\leqq\|x\|$, $x\in R$. So the Gelfand representation is norm-decreasing and hence continuous.

Lemma 2.5. *Let R be a commutative Banach algebra with a unit. If $x\in R$ is not invertible, th the set $R_x=\{wx:w\in R\}$ is a proper ideal containing x.*

Lemma 2.6. *An ideal M in R is maximal if and only if it is the kernel of a nonzero multiplic tive linear functional.*

Theorem 2.7. *For each $x\in R, \sigma(x)=\{\hat{x}(\phi):\phi\in\mathfrak{M}\}$. Hence $r_\sigma(x)=\sup_{\phi\in\mathfrak{M}}|\hat{x}(\phi)|=\|\hat{x}\|_\infty$.*

Proof. If $\lambda\in\sigma(x)$, the $\lambda e-x$ is contained in a proper ideal (Lemma 2.5), which in turn is some maximal ideal. It follows from Lemma 2.6 that $\lambda e-x$ is in the kernel of some $\phi\in\mathfrak{M}$; that $0=\phi(\lambda e-x)=\lambda-\phi(x)=\lambda-\hat{x}(\phi)$.

This shows that $\sigma(x)\subset\{\hat{x}(\phi):\phi\in\mathfrak{M}\}$. Containment in the other direction is clear from $\phi(x)\in$ $\sigma(x)$.

Definition 2.8. The *radical of R* is the intersection of all the maximal ideals of R. If the radic of R is $\{0\}$, then R is said to be semisimple.

Theorem 2.9. *The following statements are equivalent for an element f in a Banach algebra R.*

(a) *f belongs to the radical.*

(b) *the spectrum of f consists of the one point $\lambda=0$.*

(c) *for every complex number μ, the sequence $\{(\mu f)^n\}$ converge to zero.*

(d) $\hat{x}(\phi)=0$ *for all* $\phi\in\mathfrak{M}$: *that is* $\hat{x}=0\in\mathfrak{M}$.

(e) $\lim_{n\to\infty}\|x^n\|^{1/n}=0$.

Proof. It follows from Lemma 2.4. and Theorem 2.7.

3. Main Result

The main features of the Gelfand representation are summarized in the following theorem.

Theorem 2.10. (Gelfand representation theorem) *Let R be a commutative Banach algebra with unit e and $\mathfrak{M}$ be its carrier space. The Gelfand representation $\Omega : x \to \hat{x}$ has the following properties.*

(a) *Ω is a homomorphism of R onto a subalgebra $\hat{R}$ of $C(\mathfrak{M})$ and continuous.*

(b) *$\hat{e}(\phi)=1$ for all $\phi \in \mathfrak{M}$.*

(c) *$\hat{R}$ contains the constant functions and separates the point of $\mathfrak{M}$.*

(d) *$\hat{x}$ is invertible in $C(\mathfrak{M})$ if and only if x is invertible in R.*

(e) $\|\hat{x}\|_\infty = \lim_{n\to\infty} \|x^n\|^{1/n}$.

(f) *$\hat{R}$ is isomorphic to R if and only if R is semisimple.*

Proof. (a) The mapping Ω is obviously linear and multiplicative since if $f, g \in R$ and $M \in \mathfrak{M}$, $(f+g)(M)=f(M)+g(M)$ and $(fg)(M)=f(M)g(M)$. Suppose now that $f_n \to f$, that is, $\|f_n - f\| \to 0$. We have that $f_n \to f$ since $\|f_n - f\|_\infty \leqq \|f_n - f\|$. Hence the mapping is continuous.

(b) It is trivial.

(c) For $\lambda \in C$, $(\lambda e)(M)=\lambda e(M)=\lambda$ for all $M \in \mathfrak{M}$. Thus $\hat{R}$ contains the constant functions. Also, if $f(M_1)=f(M_2)$ for all $f \in R$, and hence $M_1=M_2$.

(d) A function $\hat{x} \in C(\mathfrak{M})$ is invertible in $C(\mathfrak{M})$ if and only if $f(M) \neq 0$ for all $M \in \mathfrak{M}$ by Theorem 2.7. This happens if and only if $0 \notin \sigma(f)$, that is, if and only if f is invertible in R.

(e) It follows from Theorem 2.7 and $r_\sigma(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

(f) Theorem 2.9. shows that the kernel of the Gelfand mapping is the radical of R.

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