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A Note on Open Mapping Theorem

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A topological vector space X is an *F-space* if its topology is induced by a complete invariant metric. In this paper we study open mapping theorems on various F-spaces.

To prove the main theorems, the following lemmas are necessary.

Lemma 1. (Baire's category theorem) *A complete metric space is not a union of a countable collection of nowhere dense sets.*

Proof. see [1]

Lemma 2. (The open mapping theorem) *Suppose*

a) *X is an F-space,*

b) *Y is a topological vector space,*

c) *$T: X \to Y$ is continuous and linear, and*

d) *$T(X)$ is of the second category in Y.*

Then

i) $T(X)=Y$

ii) *T is an open mapping, and*

iii) *Y is an F-space.*

Proof. see [3]

Lemma 3. *Let W be a topological vector space and $W_1 \subset W$ be a subspace of W. Let T be a continuous linear map from W onto an F-space X. If the restriction T_1 of T on W_1 is an open map, then T is also open.*

Proof. Let U be open in W. It suffices to prove that $T(U)$ is open in X. Let $x \in T(U)$. Since T_1 is onto, there exists $w \in W_1 \cap U$ such that $T_1 w = Tw = x$. Since T is open and $W_1 \cap U$ is open in W_1, $T(W_1 \cap U)$ is open in X satisfying $x \in T_1(W_1 \cap U) \subset T(U)$. Therefore $T(U)$ is open.

Theorem 4. *Let V_n, $n=1,2,3,\cdots$ be F-spaces over Φ. Let W be topological vector space and for each n, let T_n is continuously linear map from V_n into W. If $W=\bigcup_{n=1}^{\infty} T_n(V_n)$ then every continuous linear map T from W onto any F-space X is open.*

Proof. Let T be a continuous linear map from W onto X. Let V be any open set in W. We need to prove that $T(V)$ is an open set in X. It is easy to see that $\bigcup_{n=1}^{\infty} T(T_n(V_n))=X$. If $T(T_n(V_n))$ is of the first category in F for all $n \in N$, then $\bigcup_{n=1}^{\infty} T(T_n(V_n))$ itself is again of the first category in X. But by lemma 1, F-space X is not of the first category. Thus for some n_0, $T(T_{n_0}(V_{n_0}))$ is of the second category in X. Then by lemma 2, $T(T_{n_0}(V_{n_0}))=X$ *i.e.*, $T \circ T_n$ is a continuous linear map from V_{n_0} onto X.

Note that $T_{n_0}(V_{n_0})$ is a subspace of W. Now we claim that the restriction T' of T on $T_{n_0}(V_{n_0})$ is an open map.

Let U be an open subset of $T_{n_0}(V_{n_0})$. Then there exists an open set V in W such that $U=V\cap T_{n_0}(V_{n_0})$. Since T_{n_0} is continuous, $Tn_{n_0}^{-1}(U)$ is open in V_{n_0}. Since $T\circ T_{n_0}$ is open by lemma 2, $T\circ T_{n_0}(T_{n_0}^{-1}(U))=T(V\cap T_{n_0}(V_{n_0}))=T'(U)$ is open in X.

Therefore T' is an open map. By lemma 3, the proof is completed.

Theorem 5. *Suppose that W is a topological vector space. Suppose there exists a sequence W_n of subspaces such that Wn are F-spaces and $\bigcup_{n=1}^{\infty} W_n=W$. Let T be continuous linear map from W onto any F-space X. Then T is an open map.*

Proof. Let V be any open set in W. Let $T_n=T|W_n$, restriction of T on W_n. Then $\bigcup_{n=1}^{\infty} T_n(W_n)=X$. Since X is of the second category, at least one of $T_n(W_n)$ is of the second category in X, say $T_{n_0}(W_{n_0})$. Since W_{n_0} is F-space, by lemma 2, $T_{n_0}(W_{n_0})=X$ and T_{n_0} is an open map. Therefore by lemma 3, We complete the proof.

References

1. H.L. Royden, *Real Analysis.* 2-nd edition, Mamillan, 1968.
2. Ronald Larsen, *Functional Analysis.* Dekker, 1973.
3. Walter Rudin, *Functional Analysis.* McGraw-Hill Inc., 1973.