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A Common Fixed Point Theorem for Generalized Meir-Keeler type Contractive Conditions in the Saks Spaces.

By Kwon Bai Moon

1. Introduction.

Recently, some common fixed point theorems for commuting mappings in a space with two metric are proved by K. Iseki [1] and S.L. Singh [2].

In this paper, applying some common fixed point theorems of Park [3] and Park-Rhoades [4] for commuting mappings in complete metric space, we shall prove a common fixed point theorem in the Saks spaces.

2. Preliminaries.

Let X be a normed linear space. A real valued function $\|\cdot\|$ defined on X will be called a *B-norm* if it satisfies the following conditions:

(1) $\|x\|=0$ if and only if $x=0$

(2) $\|x+y\|\leq\|x\|+\|y\|$

(3) $\|\alpha x\|=|\alpha|\,\|x\|$, α : any real number.

Each real valued function $\|\cdot\|$ satisfying the above conditions (1), (2) and the following one:

(4) if the sequence $\{\alpha_n\}$ of real numbers converges to a real number α and $\|x_n-x\|\to 0$ as $n\to\infty$, then $\|\alpha_n x_n-\alpha x\|\to 0$ as $n\to\infty$, will be called a *F-norm.*

A *two norm space* is a linear space X with two norms, a B-norm $\|\cdot\|_1$ and a F-norm $\|\cdot\|_2$ and denoted by $(X, \|\cdot\|_1, \|\cdot\|_2)$.

If two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are defined on X and $x_n\in X$, $\|x_n\|_1\to 0$ as $n\to\infty$ implies $\|x_n\|_2\to 0$, then the norm $\|\cdot\|_1$ is called *non-weaker than* $\|\cdot\|_2$ in X. (denoted by $\|\cdot\|_2\leq\|\cdot\|_1$). If $\|\cdot\|_2\leq\|\cdot\|_1$ and $\|\cdot\|_1\leq\|\cdot\|_2$, then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent.*

A sequence $\{x_n\}$ of points in a two norm space $(X, \|\cdot\|_1, \|\cdot\|_2)$ is said to be *γ-convergent* to x_0 in X if $\sup\|x_n\|_1<\infty$ and $\lim_{n\to\infty}\|x_n-x_0\|_2=0$, and a sequence $\{x_n\}$ in a two-norm space is said to be *γ-cauchy* if $(x_{p_n}-x_{q_n}\xrightarrow{\gamma}0$ as $p_n, q_n\to\infty$.

A two norm space is called *γ-complete* if for every γ-Cauchy sequence $\{x_n\}$ in two norm space, there exists $x_0\in X$ such that $x_n\xrightarrow{\gamma}x_0$.

Let X be a normed linear space cond $\|\cdot\|_1$ is a B-norm and $\|\cdot\|_2$ is a F-norm on X. Let $X_s=\{x\in X : \|x\|_1<1\}$ and define $d(x,y)=\|x-y\|_2$ for all x,y in X_s. Then d is a metric on X_s and the metric space (X_s, d) will be called a *Saks set.* If (X_s, d) is complete, it will be called a *Saks space* and denoted (X_s, d) by $(X, \|\cdot\|_1, \|\cdot\|_2)$.

In [5], W. Orlicz has proved the following:

Theorem 1. *Let $(X_s, d)=(X, \|\cdot\|_1, \|\cdot\|_2)$ be a Saks space. Then the following statements are equivalent:*

(1) $\|\cdot\|_1$ *is equivalent to* $\|\cdot\|_2$ *on* X.

(2) $(X, \|\cdot\|_1)$ *is a Banch spaace and* $\|\cdot\|_2 \geq \|\cdot\|_1$ *on* X.

(3) $(X, \|\cdot\|_2)$ *is a Frechet space and* $\|\cdot\|_1 \geq \|\cdot\|_2$ *on* X.

3. Main Theorem.

Let f be a continuous selfmap of a Saks space $(X_s, d)=(X, \|\cdot\|_1, \|\cdot\|_2)$, $C_f=\{g \mid g: X \to X$ such that $fg=gf$ and $gX \subset fX\}$.

For $x_0 \in X$, the sequence $\{fx_n\}_{n=1}^{\infty}$ is called the *f-iteration of* x_0 *under* g, and is defined by $fx_n = gx_{n-1}$, $n=0, 1, 2, \cdots$ with the understanding that, if $fx_n = fx_{n+1}$ for some n, then $fx_{n+j}=fx_n$ for each $j \geq 0$.

Theorem 2. *Let $(X_s, d)=(X, \|\cdot\|_1, \|\cdot\|_2)$ be a Saks space which $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ on X. If two commuting mappings f and g from X into itself satisfy the following conditions:*

(1) $g \in C_f$

(2) *f and g are continuous with respect to* $\|\cdot\|_1$.

(3) *For each $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$\varepsilon \leq \max\{\|fx-fy\|_2, \|fx-gx\|_2, \|fy-gy\|_2, [\|fx-gy\|_2+\|fy-gx\|_2]/2\} < \varepsilon+\delta$ *implies* $\|gx-gy\|_2 < \varepsilon$.

Then f and g have a unique common fixed point p in X, and for each $x_0 \in X$, any f-iteration of x_0 under g converges to some $\xi \in X$ satisfying $f\xi = p$.

Lemma 3. *Let f and g satisfy the hypotheses of Theorem 2. Then* $r=\inf\{\|fx_n - fx_{n+1}\|_2 \mid n=1, 2, \cdots\}=0$.

Lemma 4. *Let f and g be as in Theorem 2. If there exists a $\xi \in X$ such that $f\xi = g\xi$, then $f\xi$ is the unique common fixed point of f and g in X.*

Proof of Theorem 2. From Lemma 4 it is sufficient to find a point ξ such that $f\xi = g\xi$.

Let $x_0 \in X$ and let $\{fx_n\}$ be an f-iteration of x_0 under g. By Lemma 3, $r=0$. Then we have the following two cases (a) and (b).

(a) If there exists an n such that $fx_n = fx_{n+1}$, then $fx_{n+1}=gx_n=fx$ and we are finished.

(b) Assume $\|fx_n - fx_{n+1}\|_2 \neq 0$ for each n. We wish to show that $\{fx_n\}$ is Cauchy. Suppose not. Then there exists an $\varepsilon > 0$ and a subsequence $\{fx_{n_i}\}$ of $\{fx_n\}$ such that $\|fx_{n_i} - fx_{n_{i+1}}\|_2 > 2\varepsilon$. From (3), there exists a δ satisfying $0 < \delta < \varepsilon$ for which (3) is true. Since $r=0$, there exists an N such that $m > N$ implies $\|fx_m - fx_{m+1}\|_2 < \varepsilon/6$. Let $n_j \geq N$. We shall now show that there exists an integer j satisfying $n_i < j < n_{i+1}$ such that

(4) $\varepsilon+\delta/3 \leq \|fx_{n_i} - fx_j\|_2 < \varepsilon + 2\delta/3$.

First of all there exists value of j such that $\|fx_{n_i} - fx_j\|_2 \geq \varepsilon+\delta/3$. For $\|fx_{n_i} - fx_j\|_2 < \varepsilon+\delta/3$. For example, choose $j=n_i+1$ and $j=n_i+2$.

Pick j to be the smallest integer greater than n_i such that $\|fx_{n_i} - fx_j\|_2 \geq \varepsilon+\delta/3$. Then $\|fx_{n_i} - fx_{j-1}\|_2 < \varepsilon+\delta/3$ and $\|fx_{n_i} - fx_j\|_2 \leq \|fx_{n_i} - fx_{j-1}\|_2 + \|fx_{j-1} - fx_j\|_2 < \varepsilon+\delta/3+\delta/6 < \varepsilon+2\delta/3$ and (4) is established. Therefore from (3), $\|gx_{n_i} - gx_j\|_2 < \varepsilon$; *i.e.*, $\|fx_{n_i+1} - fx_{j+1}\|_2 < \varepsilon$. On the other hand, $\|fx_{n_i} - fx_j\|_2 \leq \|fx_{n_i} - fx_{n_i+1}\|_2 + \|fx_{n_i+1} - fx_{j+1}\|_2 + \|fx_{j+1} - fx_j\|_2 < \delta/6+\varepsilon+\delta/6 = \varepsilon+\delta/3$, contradicting (4).

Therefore $\{fx_n\}$ is a Cauchy sequence with respect to $\|\cdot\|_2$. Since $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$, $\{fx_n\}$ is Cauchy sequence with respect to $\|\cdot\|_1$ and from Theorem 1, since $(X, \|\cdot\|_1)$ is a Banach space, there exists $\xi \in X$ such that $\|fx_n - \xi\|_1 \to 0$ as $n \to \infty$, and also in view of $fx_{n+1} = gx_n$. We have $\|gx_n - \xi\|_1 \to 0$ as $n \to \infty$. Since f and g are continuous with respect to $\|\cdot\|_1$, we have $\|gfx_n - g\xi\|_1 \to 0$ and $\|fgx_n - f\xi\|_1 \to 0$ as $n \to \infty$. But since $fg = gf$, $g\xi = f\xi = p$. By Lemma 4, p is the unique common fixed point of f and g. And we have the required conclusion.

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