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On certain almost Kaehlerian manifolds

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I. Introduction

Recently many authors have studied on Kaehlerian manifolds with parallel or vanishing Bochner curvature tensor.

The main purpose of this paper is to investigate the properties of almost Kaehlerian manifold with vanishing Bochner curvature tensor. In II, we introduce almost Kaehlerian manifold. In III, we suppose that almost Kaehlerian manifolds have vanishing Bochner curvature tensor. Under this assumption, we got the following theorems.

Theorem A. *An almost Kaehlerian manifold with vanishing Bochner curvature tensor is a Kaehlerian manifold.*

Theorem B. *Let M be an almost Kaehlerian manifold with vanishing Bochner curvature tensor. Then the following statements are equivalent.*

(1) *M has the constant scalar curvature.*

(2) *M has the parallel Ricci tensor.*

(3) *M is locally symmetric.*

II. Preliminaries

Let M be $2n$-dimensional almost Kaehlerian manifold covered by a system of coordinate neighbourhoods $\{U;x^h\}$, where here and in the sequel the indices $h,i,j,k,\cdots$ run over the range $1,2,\cdots,2n$ and let $(F_j{}^i, g_{ji})$ be the almost Kaehlerian structure, that is,

$$F_j{}^kF_k{}^i=-\delta_j^i, \tag{2.1}$$

and g_{ji} a Riemannian metric such that

$$g_{kl}F_j{}^kF_i{}^l=g_{ji}, \tag{2.2}$$

and

$$\nabla_jF_{ih}+\nabla_iF_{hj}+\nabla_hF_{ji}=0, \tag{2.3}$$

where we denote by $\left\{ {h \atop ji} \right\}$ and ∇_j the Christoffel symbols formed with g_{ji} and the operator of covariant differentiation with respect to $\left\{ {h \atop ji} \right\}$ respectively.

From (2.2), we have

$$F_{ji}=-F_{ij}. \tag{2.4}$$

Now we denote by $K_{kji}{}^h$, K_{ji} and K local components of the curvature tensor, the Ricci tensor, the scalar curvature of M respectively.

The Bochner curvature tensor of M is defined to be

(2.5) $$B_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} + F_k{}^h M_{ji} - F_j{}^h M_{ki} + M_k{}^h F_{ji} - M_j{}^h F_{ki} - 2(M_{kj} F_i{}^h + F_{kj} M_i{}^h),$$

where

$$L_{ji} = -\frac{1}{(n+4)} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji},\ L_k{}^h = L_{kt} g^{th},$$

$$M_{ji} = -L_{jt} F_i{}^t = -\frac{1}{(n+4)} H_{ji} + \frac{1}{2(n+2)(n+4)} K F_{ji},$$

$$M_k{}^h = M_{kt} g^{th}, \qquad H_{ji} = -K_{jt} F_i{}^t.$$

We define F_{jih} by

(2.6) $$F_{jih} = \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji}.$$

Transvecting (2.6) with F^{ih}, and then using (2.1), (2.3) and (2.4), we get

(2.7) $$F_{jih} F^{ih} = -2 F_j{}^i (\nabla_h F_i{}^h),$$

and transvecting (2.7) with $F_k{}^j$,

(2.8) $$\nabla_h F_k{}^h = 0.$$

By virture of (2.3), (2.6) and (2.8), we can see that the rotation and divergence of skew symmetric tensor F vanish. Thus we have

Lemma. *In an almost Kaehlerian manifold, the structure tensor F_{ji} is harmonic.*

III. The main results

In this section, we assume that the almost Kaehlerian manifold is of vanishing Bochner curvature tensor. From (2.1), we find easily that $F_{ji}F^{ji}$ is constant function on M.

Taking the Laplacian operator Δ into the constant function $F^{ji}F_{ji}$ on M, we find

(3.1) $$0 = \frac{1}{2}\Delta(F^{ji}F_{ji}) = g^{kj}(\nabla_k \nabla_j F_{ih}) F^{ih} + \|\nabla_j F_{ih}\|^2,$$

from which, using (2.3) we obtain

(3.2) $$g^{kj}(\nabla_k \nabla_i F_{hj} + \nabla_k \nabla_h F_{ji}) F^{ih} - \|\nabla_j F_{ih}\|^2 = 0,$$

or, using Ricci's identity,

(3.3) $$g^{kj}(K_{kih}{}^t F_{tj} + K_{kij}{}^t F_{ht} - \nabla_i \nabla_k F_{hj}) F^{ih} + g^{kj}(K_{khj}{}^t F_{ti} + K_{khi}{}^t F_{jt} - \nabla_h \nabla_k F_{ji}) F^{ih} + \|\nabla_j F_{ih}\|^2 = 0,$$

which reduces to

(3.4) $$2(K_{kih}{}^t F_t{}^k - K_{it} F_h{}^t) F^{ih} = -\|\nabla_j F_{ih}\|^2.$$

Differentiating (2.1) covariantly, we have

(3.5) $$(\nabla_k F_{hj}) F^{ih} + F_{hj} (\nabla_k F^{ih}) = 0.$$

Again differentiating (3.5) covariantly, we get

(3.6) $$(\nabla_i \nabla_k F_{hj}) F^{ih} + (\nabla_i F_{hj})(\nabla_k F^{ih}) + F_{hj}(\nabla_i \nabla_k F^{ih}) = 0.$$

Transvecting (3.6) with g^{kj} and using (2.8), we obtain

(3.7) $$(\nabla_j \nabla_i F_h{}^j) F^{ih} + (\nabla_i F_{hj})(\nabla^j F^{ih}) = 0,$$

or, using Ricci's identity, (2.3) and (2.8),

(3.8) $$(K_{kih}{}^t F_t{}^k - K_{it} F_h{}^t) F^{ih} + (\nabla^j F^{ih})(\nabla_h F_{ji}) + \|\nabla_j F_{ih}\|^2 = 0.$$

Substituting (3.8) into (3.4), we have

(3.9) $$(K_{kih}{}^t F_t{}^k - K_{it} F_h{}^t) F^{ih} = (\nabla^j F^{ih})(\nabla_h F_{ji}),$$

substituting this into (3.8),

(3.10) $$\|\nabla_j F_{ih}\|^2 = -2(\nabla^j F^{ih})(\nabla_h F_{ji}),$$

which and (3.9) imply

$$\|\nabla_j F_{ih}\|^2 = -2(K_{kih}{}^t F_t{}^k - K_{it} F_h{}^t) F^{ih}. \tag{3.11}$$

From the hyphothesis of this section, we get

$$K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} + F_k{}^h M_{ji} - F_j{}^h M_{ki} + M_k{}^h F_{ji} - M_j{}^h F_{ki} - 2(M_{kj} F_i{}^h + F_{kj} M_i{}^h) = 0. \tag{3.12}$$

From this and (3.11), we have

$$\begin{aligned}\|\nabla_j F_{ih}\|^2 = 2F^{ih}[\{&-nL_{it} + \delta_i^k L_{kt} - L_k{}^k g_{it} + L_i{}^k g_{kt} + F_i{}^k M_{kt} + M_i{}^k F_{kt} + 2(M_{ki} F_t{}^k \\ &+ F_{ki} M_t{}^k)\} F_h{}^t - \{-{}^k{}_t L_{ih} + {}^i{}_t L_{kh} - L_k{}^t g_{ih} + L_i{}^t g_{kh} - F_k{}^t M_{ih} + F_i{}^t M_{kh} \\ &- M_k{}^t F_{ih} + M_i{}^t F_{kh} + 2(M_{ki} F_h{}^t + F_{ki} M_h{}^t)\} F_t{}^k],\end{aligned} \tag{3.13}$$

which reduces to

$$\|\nabla_j F_{ih}\|^2 = 2[(2n-4) L_t{}^t - (2n-4) M_{st} F^{st}]. \tag{3.14}$$

Moreover $M_{st} F^{st} = -L_{su} F_t{}^u F^{st} = L_t{}^t$. Hence, from (3.14) and the above equation, we obtain

$$\nabla_j F_{ih} = 0. \tag{3.15}$$

Thus we have

Theorem 3.1. *Almost Kaehlerian manifold with vanishing Bochner curvature tensor is a Kaehlerian manifold.*

Differentiating (3.12) covariantly by using (2.5), we have

$$\begin{aligned}\nabla_l K_{kji}{}^h = &\frac{1}{(n+4)}(\delta_k^h \nabla_l K_{ji} - \delta_j^h \nabla_l K_{ki} + g_{ji} \nabla_l K_k{}^h - g_{ki} \nabla_l K_j{}^h) \\ &- \frac{1}{(n+2)(n+4)}(\nabla_l K)(g_{ji}\delta_k^h - g_{ki}\delta_j^h + F_k{}^h F_{ji} - F_j{}^h F_{ki} - 2F_i{}^h F_{kj}) \\ &- \frac{1}{(n+4)}[F_k{}^h F_i{}^t(\nabla_l K_{jt}) - F_j{}^h F_i{}^t(\nabla_l K_{kt}) + F_{ji} F^{ht}(\nabla_l K_{kt}) \\ &- F_{ki} F^{ht}(\nabla_l K_{jt}) - 2F_i{}^h F_j{}^t(\nabla_l K_{kt}) - 2F_{kj} F^{ht}(\nabla_l K_{it})].\end{aligned} \tag{3.16}$$

From the above equation, we get

Proposition 3.2. *If the Ricci tensor of almost Kaehlerian manifold M with vanishing Bochner curvature tensor is parallel, then M is locally symmetric.*

We now assume that the scalar curvature is constant. By a straightforward computation from (2.5), we can prove $\nabla_t B_{kji}{}^t = -n(\nabla_k L_{ji} - \nabla_j L_{ki})$. By virtue of this equation and the definition of L_{ji}, we have

$$\nabla_l K_{ji} - \nabla_j K_{li} = 0, \tag{3.17}$$

that is, $\nabla_l K_{ji}$ is a symmetric tensor. From (3.15) and the Ricci identity $\nabla_k \nabla_j F_i{}^h - \nabla_j \nabla_k F_i{}^h = K_{kjs}{}^h F_i{}^s - K_{kji}{}^s F_s{}^h$, we get

$$K_{kji}{}^s F_s{}^h = K_{kjs}{}^h F_i{}^s. \tag{3.18}$$

Transvecting (3.18) with g^{ji}, we obtain

$$K_k{}^s F_s{}^h = K_{ksr}{}^h F^{sr} = \frac{1}{2}(K_{ksr}{}^h - K_{krs}{}^h) F^{sr}, \tag{3.19}$$

or,

$$K_k{}^s F_s{}^h = -\frac{1}{2} K_{srk}{}^h F^{sr}, \tag{3.20}$$

from which

$$K_k{}^s F_s{}^h - K_s{}^h F_k{}^s = 0, \tag{3.21}$$

from which again

$$K_{ji}=F_j{}^t F_i{}^s K_{ts}. \tag{3.22}$$

Using (3.16) and Bianchi identity $\nabla_l K_{kji}{}^h+\nabla_k K_{jli}{}^h+\nabla_j K_{lki}{}^h=0$ and (3.17), we get

$$F_i{}^h F_j{}^t(\nabla_l K_{kt})+F_l{}^t(\nabla_k K_{jt})+F_i{}^h F_k{}^t(\nabla_j K_{lt})+F_{kj}F^{ht}(\nabla_l K_{it})+F_{jl}F^{ht}(\nabla_k K_{it}) \\ +F_{lk}F^{ht}(\nabla_j K_{it})=0. \tag{3.23}$$

Contracting h and k at (3.23), we have

$$F_i{}^k F_j{}^t(\nabla_l K_{kt})+F_i{}^k F_l{}^t(\nabla_k K_{jt})-\nabla_j K_{li}+\nabla_l K_{ij}+F_{jl}F^{kt}(\nabla_k K_{it})-\nabla_j K_{il}=0. \tag{3.24}$$

Using (3.17) and (3.22), we obtain from (3.24)

$$\nabla_l K_{ji}=0, \tag{3.25}$$

that is, Ricci tensor is parallel.

From proposition 3.2 and the above fact, we have

Theorem 3.3. *Let M be an almost Kaehlerian manifold with vanishing Bochner curvature tenso Then the following statements are equivalent*

(1) *M has the constant scalar curvature;*

(2) *M has the parallel Ricci tensor;*

(3) *M is locally symmetric.*

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