On certain almost Kaehlerian manifolds

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I. Introduction

Recently many authors have studied on Kaehlerian manifolds with parallel or vanishing Bochner curvature tensor.

The main purpose of this paper is to investigate the properties of almost Kaehlerian manifold with vanishing Bochner curvature tensor. In II, we introduce almost Kaehlerian manifold. In III, we suppose that almost Kaehlerian manifolds have vanishing Bochner curvature tensor. Under this assumption, we got the following theorems.

Theorem A. An almost Kaehlerian manifold with vanishing Bochner curvature tensor is a Kaehlerian manifold.

Theorem B. Let $M$ be an almost Kaehlerian manifold with vanishing Bochner curvature tensor. Then the following statements are equivalent.

1. $M$ has the constant scalar curvature.
2. $M$ has the parallel Ricci tensor.
3. $M$ is locally symmetric.

II. Preliminaries

Let $M$ be $2n$-dimensional almost Kaehlerian manifold covered by a system of coordinate neighbourhoods $\{U; x^i\}$, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $1, 2, \cdots, 2n$ and let $\langle F_j, g_{ji} \rangle$ be the almost Kaehlerian structure, that is,

$$F_j = -\delta_j,$$

and $g_{ji}$ a Riemannian metric such that

$$g_{kl} F_k F_l = g_{ji},$$

and

$$\nabla_j F_k = \nabla_k F_j = \nabla_l F_l = 0,$$

where we denote by $\left\{ \frac{h}{j_i} \right\}$ and $\nabla_j$ the Christoffel symbols formed with $g_{ji}$ and the operator of covariant differentiation with respect to $\left\{ \frac{h}{j_i} \right\}$ respectively.

From (2.2), we have

$$F_{ji} = -F_{ij}.$$ 

Now we denote by $K_{hji}, K_{ij}$ and $K$ local components of the curvature tensor, the Ricci tensor, the scalar curvature of $M$ respectively.

The Bochner curvature tensor of $M$ is defined to be
\[ B_{kl}^b = K_{kl}^b + \delta_k^b L_{j} - \delta_l^b L_{ki} + L_k^b g_{j} + L_j^b g_{ki} + F_k^b M_{j} - F_l^b M_{ki} + M_k^b F_{j} - M_l^b F_{ki} - 2(M_k^b F_i^b + F_k^b M_i^b), \]

where

\[ L_{ji} = - \frac{1}{(n+4)} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji}, \]
\[ M_{jl} = - L_{jl} F_i^l = - \frac{1}{(n+4)} H_{ji} + \frac{1}{2(n+2)(n+4)} K F_{ji}, \]
\[ M_k^b = M_{ki} g^{kb}, \]
\[ H_{ji} = - K_{ji} F_i^b. \]

We define \( F_{ij} \) by

\[ F_{ij} = \varphi_j F_{ih} + \varphi_i F_{hj} + \varphi_h F_{ji}. \]

Transvecting (2.6) with \( F^{ik} \), and then using (2.1), (2.3) and (2.4), we get

\[ F_{jik} F^{ik} = - 2 F_{ji} (\varphi_k F^k) \]

and transvecting (2.7) with \( F_k^i \),

\[ \varphi_i F_k^b = 0. \]

By virtue of (2.3), (2.6) and (2.8), we can see that the rotation and divergence of skew symmetric tensor \( F \) vanish. Thus we have

**Lemma.** In an almost Kaehlerian manifold, the structure tensor \( F_{ij} \) is harmonic.

**III. The main results**

In this section, we assume that the almost Kaehlerian manifold is of vanishing Bochner curvature tensor. From (2.1), we find easily that \( F_{ij} F^{ij} \) is a constant function on \( M \).

Taking the Laplacian operator \( \Delta \) into the constant function \( F_{ij} F_{ji} \) on \( M \), we find

\[ 0 = \frac{1}{2} \Delta (F_{ij} F_{ji}) = g^{bi} (\varphi_i F_{ih}) F^{ik} + \| \varphi_i F_{ih} \|^2, \]

from which, using (2.3) we obtain

\[ g^{bi} (\varphi_i F_{ih} + \varphi_h F_{ji}) F^{ik} - \| \varphi_i F_{ih} \|^2 = 0, \]

or, using Ricci's identity,

\[ g^{bi} (K_{bih} F_{ij} + K_{bji} F_{hi} - \varphi_i F_{bh}) F^{ik} + g^{bi} (K_{bih} F_{ji} + K_{bji} F_{hi} - \varphi_j F_{bh}) F^{ik} + \| \varphi_i F_{ih} \|^2 = 0, \]

which reduces to

\[ 2 (K_{bih} F_{jih} - K_{bji} F_{hij}) F^{ik} = - \| \varphi_j F_{ih} \|^2. \]

Differentiating (2.1) covariantly, we have

\[ (\varphi_i F_{hi}) F^{ik} + F_{hi} (\varphi_k F^{ik}) = 0. \]

Again differentiating (3.5) covariantly, we get

\[ (\varphi_i F_{hi}) F^{ik} + (\varphi_i F_{hi}) (\varphi_k F^{ik}) + F_{hi} (\varphi_k F^{ik}) = 0. \]

Transvecting (3.6) with \( g^{bi} \) and using (2.8), we obtain

\[ (\varphi_i F_{hi}) F^{ik} + (\varphi_i F_{hi}) (\varphi_j F_{ik}) = 0, \]

or, using Ricci’s identity, (2.3) and (2.8),

\[ (K_{bih} F_{jih} - K_{bji} F_{hij}) F^{ik} + (\varphi_j F_{ih}) (\varphi_k F_{ih}) + \| \varphi_j F_{ih} \|^2 = 0. \]

Substituting (3.8) into (3.4), we have

\[ (K_{bih} F_{jih} - K_{bji} F_{hij}) F^{ik} = (\varphi_j F_{ih}) (\varphi_k F_{ih}), \]

substituting this into (3.8),

\[ \| \varphi_j F_{ih} \|^2 = -2 (\varphi_j F_{ih}) (\varphi_k F_{ih}), \]

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which and (3.9) imply

\[(3.11)\quad \|\varphi_j F_{ih}\|^2 = -2\langle K_{bih} F^b - K_{ih} F^i \rangle F^{ih}.\]

From the hypothesis of this section, we get

\[(3.12)\quad K_{bih} + (\delta_i^b L_{ji} - \delta_j^b L_{hi} + L^b g_{ji} - L^j g_{hi} + F^i j M_{hi} - F^j i M_{hi} + M^i b F_{ji} - M^j b F_{hi})
- \langle F_{ih} F^i \rangle = 0.
\]

From this and (3.11), we have

\[(3.13)\quad \|\varphi_j F_{ih}\|^2 = 2 F^{ih} (\{ -n L_{ih} + (\delta_i^b L_{hi} - \delta_j^b L_{hi} + L^b g_{ji} + L_j^i g_{hi} + F^i j M_{hi} + M^i b F_{hi} + 2 \langle M_i F^b \rangle
+F_{ih} M^b \} F_i - \langle -n L_{ih} + (\delta_i^b L_{hi} - \delta_j^b L_{hi} + L^b g_{ji} + L_j^i g_{hi} + F^i j M_{hi} + M^i b F_{hi} + F_i M_{hi} \rangle F_i \})
-M_i F_{ih} + M_i F_{ih} + 2 \langle M_i F^i + F_{ih} M^b \} F_i \rangle,
\]

which reduces to

\[(3.14)\quad \|\varphi_j F_{ih}\|^2 = 2 (2n - 4) L_i - (2n - 4) M_{ih} F^{ii}.
\]

Moreover \( M_{ih} F^{ii} = -L_{ih} F_{ih} F^{ii} = L_i. \) Hence, from (3.14) and the above equation, we obtain

\[(3.15)\quad \varphi_j F_{ih} = 0.
\]

Thus we have

**Theorem 3.1.** *Almost Kaehlerian manifold with vanishing Bochner curvature tensor is a Kaehlerian manifold.*

Differentiating (3.12) covariantly by using (2.5), we have

\[(3.16)\quad \varphi_i K_{jij} = \frac{1}{(n+4)} (\delta_i^b \varphi_j K_{bj} - \delta_j^b \varphi_i K_{bj} + g_{ji} \varphi_i K_{bj} - g_{ij} \varphi_i K_{bj})
- \frac{1}{(n+4)} (\varphi_i K) (g_{ji} \delta_i^b - g_{ij} \delta_j^b + F^i j F_{bj} - F^j i F_{bj} - 2 F_i^b F_{bj})
- \frac{1}{(n+4)} (F_i^b F^j (\varphi_j K_{bj}) - F_j^b F^i (\varphi_i K_{bj}) + F^i j F_{bj} (\varphi_i K_{bj}))
- F_i^b F_{bj} (\varphi_i K_{bj}) - 2 F_i^b F_j (\varphi_i K_{bj}) - 2 F_i^b F_{bj} (\varphi_i K_{bj}).
\]

From the above equation, we get

**Proposition 3.2.** *If the Ricci tensor of almost Kaehlerian manifold \( M \) with vanishing Bochner curvature tensor is parallel, then \( M \) is locally symmetric.*

We now assume that the scalar curvature is constant. By a straightforward computation from (2.5), we can prove \( \varphi_i B_{jij} = -n (\varphi_j L_{ij} - \varphi_i L_{hi}). \) By virtue of this equation and the definition of \( L_{ij}, \) we have

\[(3.17)\quad \varphi_i K_{jij} - \varphi_i K_{ij} = 0,
\]

that is, \( \varphi_i K_{jij} \) is a symmetric tensor. From (3.15) and the Ricci identity \( \varphi_i \varphi_j F^b - \varphi_i \varphi_j F^b = K_{bji}^i F_j - K_{bji}^i F_j, \) we get

\[(3.18)\quad K_{bji}^i F_j = K_{bji}^i F_j.
\]

Transvecting (3.18) with \( g^{ii}, \) we obtain

\[(3.19)\quad K_{bji}^i F_j = K_{bji}^i F_j = \frac{1}{2} \langle K_{bji} - K_{bji} \rangle F^{ij},
\]

or,

\[(3.20)\quad K_{bji}^i F_j = -\frac{1}{2} K_{bji}^i F_j,
\]

from which

\[(3.21)\quad K_{bji}^i F_j - K_{bji}^i F_j = 0,
\]

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from which again
\[(3.22) \quad K_{ji} = F_i F_j K_{ij}.\]

Using (3.16) and Bianchi identity \(\varphi K_{ij} = \varphi [K_{ij}] + \varphi K_{ij}^a = 0\) and (3.17), we get
\[(3.23) \quad F_i F_j (\varphi K_{ij}) + F_i F_j (\varphi K_{ij}) + F_j F_i (\varphi K_{ij}) + F_i F_j (\varphi K_{ij}) + F_j F_i (\varphi K_{ij}) = 0.\]

Contracting \(h\) and \(k\) at (3.23), we have
\[(3.24) \quad F_i F_j (\varphi K_{ij}) + F_i F_j (\varphi K_{ij}) - \varphi K_{ij} + \varphi K_{ij} + F_i F_j (\varphi K_{ij}) - \varphi K_{ij} = 0.\]

Using (3.17) and (3.22), we obtain from (3.24)
\[(3.25) \quad \varphi K_{ij} = 0,
\]
that is, Ricci tensor is parallel.

From proposition 3.2 and the above fact, we have

**Theorem 3.3.** Let \(M\) be an almost Kaehlerian manifold with vanishing Bochner curvature tensor. Then the following statements are equivalent

1. \(M\) has the constant scalar curvature;
2. \(M\) has the parallel Ricci tensor;
3. \(M\) is locally symmetric.

**Bibliography**