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On the BCK-Algebra.

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Abstract

(1) The direct product $\prod_{i=1}^{n} E_i$ of BCK-algebras E_i, $(i=1,2,3,\cdots,n)$, is a BCK-algebra.

(2) Let E be a BCK-algebra and $A_1, A_1, \cdots, A_n$ ideals of E. Define a mapping $f: E \to \prod_{i=1}^{n}(E/A_i)$ by the rule $f(x)=(A_1x, A_2x, \cdots, A_nx)$. Then f is a homomorphism.

Introduction and preliminaries.

In [1] Iséki describe a BCK-algebra as a general algebra $E=\langle X;*,0\rangle$ satisfying the following axioms: For any $x,y,z \in X$.

(1) $(x*y)*(x*z) \leq Z*Y$

(2) $x*(x*y) \leq y$

(3) $x \leq x$

(4) $x \leq y$, $y \leq x$ imply $x=y$

(5) $0 \leq x$

Where $x \leq y$ means $x*y=0$

Let $E=\langle X;*,0\rangle$ and $F=\langle X';*,0\rangle$ be BCK-algebras.

A mapping $f: X \to X'$ is called a homomorphism, if for any $x,y \in X$, $f(x*y)=f(x)*f(y)$.

For a homomorphism $f, f(0)=0$.

Let A be a non empty subset of BCK-algebra E.

A is called to be an ideal, if (1) $0 \in A$, (2) $x \in A$ and $y*x \in A$ imply $y \in A$.

The simplest examples of ideal are $\{0\}$ and X.

We can see that the intersection $A \cap B$ of ideals is an ideal.

Furthermore the intersection of finite number of ideals is an ideal.

We shall state some properties on BCK-algebra:

(1) $0*x=0$, $x*0=x$

(2) $X*x=0$

(3) $x*y \leq x$

(4) $x \leq y$ imply $z*y \leq z*x$

(5) $x \leq y$, $y \leq z$ imply $x \leq z$

(6) $(x*y)*z=(x*z)*y$

(7) $x*y \leq z$ imply $x*z \leq y$

(8) $x\leq y$ imply $x*z\leq y*z$

For a homomorphism $f: E\to F$ of BCK-algebras, $Ker(f)$ is an ideal of E and if B is an ideal of F, then $f^{-1}(B)$ is an ideal of E.

If A is an ideal of BCK-algebra E, then the quotient E/A is a BCK-algebra. For each homomorphism $f: E\to F$ of BCK-algebras with $f(A)\to 0$, where A is an ideal of E, there is a unique homomorphism $\phi: E/A\to F$ such that $f=\phi\circ p$, where $p: E\to E/A$ homomorphism.

For the above homomorphism $\phi: E/A\to F$, ϕ is an isormorphism if $f: E\to F$ is an epimorphism of BCK-algebras with Kernel A.

Main Theorems.

We construct the direct product of BCK-algebras. Let $E_1, E_2, \cdots, E_n$ be BCK-algebras. Their direct product $E=\prod_{i=1}^{n} E_i$ is the set of all sequence $x=(x_1, x_2, \cdots, x_n)$ with $x_i\in E_i$ $(1\leq i\leq n)$ and componentwise operation.

Theorem 1. *$\prod_{i=1}^{n} E_i$ is a BCK-algebra with constant* $(0, 0, 0, \cdots, 0)$.

Proof. For any $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$ and $(z_1, z_2, \cdots, z_n)$ of $\prod_{i=1}^{n} E_i$.

(1) $[\{(x_1, x_2, \cdots, x_n)*(y_1, y_2, \cdots, y_n)\}*\{(x_1, x_2, \cdots, x_n)*(z_1, z_2, \cdots, z_n)\}]*[(z_1, z_2, \cdots, z_n)*(y_1, y_2, \cdots, y_n)]=((x_1*y_1, \cdots, x_n*y_n)*(x_1*z_1, \cdots, x_n*z_n))*(z_1*y_1, \cdots, z_n*y_n)=((x_1*y_1)*(x_1*z_1), \cdots, (x_n*y_n)*(x_n*z_n))*(z_1*y_1, \cdots, z_n*y_n)=(((x_1*y_1)*(x_1*z_1))*(z_1*y_1), \cdots, ((x_n*y_n)*(x_n*z_n))*(z_n*y_n)=(0, 0, \cdots, 0)$.

(2) $((x_1, \cdots, x_n)*((x_1, \cdots, x_n)*(y_1, \cdots, y_n)))*(y_1, \cdots, y_n)=((x_1, \cdots, x_n)*(x_1*y_1, \cdots, x_n*y_n))*(y_1, \cdots, y_n)=(x_1*(x_1*y_1), \cdots, x_n*(x_n*y_n))*(y_1, \cdots, y_n)=((x_1*(x_1*y_1))*y_1, \cdots, (x_n*(x_n*y_n))*y_n)=(0, 0, \cdots, 0)$.

(3) $(x_1, \cdots, x_n)*(x_1, \cdots, x_n)=(x_1*x_1, \cdots, x_n*x_n)=(0, 0, \cdots, 0)$.

(4) $(x_1, x_2, \cdots, x_n)*(y_1, \cdots, y_n)=(0, 0, \cdots, 0)\Rightarrow(x_1*y_1, \cdots, x_n*y_n)=(0, 0, \cdots, 0)\Rightarrow x_1*y_1=0, \cdots, x_n y_n=0$ $\Rightarrow x_1=y_1, \cdots, x_n=y_n$ $\therefore (x_1, \cdots, x_n)=(y_1, \cdots, y_n)$. similarly $(y_1, \cdots, y_n)*(x_1, \cdots, x_n)=(0, \cdots, 0)\Rightarrow(x_1, \cdots, x_n)=(y_1, \cdots, y_n)$.

(5) $(0, \cdots, 0)*(x_1, \cdots, x_n)=(0*x_1, \cdots, 0*x_n)=(0, \cdots, 0)$.

Theorem 2. *Let E be a BCK-algebra and $A_1, \cdots, A_n$ ideals of E. Define a mapping $f: E\to\prod_{i=1}^{n}(E/A_i)$ by the rule $f(x)=(A_1x, \cdots, A_nx)$ then f is a homomorphism.*

Proof. For any $x, y\in E$,

$$f(x*y)=(A_1(x*y), \cdots, A_n(x*y))=(A_1x*A_1y, \cdots, A_nx*A_ny)$$
$$=(A_1x, \cdots, A_nx)*(A_1y, \cdots, A_ny)=f(x)*f(y)$$

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