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Note on Fredholm Operator

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I. Introduction

In this paper I will give some characterizations of the Fredholm operators. The following notations are used throughout; $\boldsymbol{C}$ denote the complex field, X a Banach space over $\boldsymbol{C}$, $B(X)$ the Banach algebra of bounded linear operators on X, $\mathrm{Inv}(B(X))$ the set of invertible operators in $B(X)$.

Definition 1. $F(X)=\{T\in B(X)\,|\dim(TX)<\infty\}$

$K(X)=\{T\in B(X)\,|\,\overline{TU}$: compact, U : closed unit ball of $X\}$

Clearly, $F(X)\subset B(X)$ is an ideal and $K(X)\subset B(X)$ is a closed ideal.

Definition 2. $T\in B(X)$ is a Fredholm opertor if

i) $\alpha(T)=\dim(\mathrm{Ker}(T))<\infty$

ii) TX is closed in X

iii) $\beta(T)=\dim(X/TX)<\infty$

The set of Fredholm operators is denoted $\Phi(X)$.

Observe that if the algebraic dimension of X/TX is finite then, by the open mapping theorem, it follows that TX is closed in X.

It follows from the Riesz theory that if $T\in K(X)$, $\lambda\neq 0$ then $\lambda-T\in\Phi(X)$.

Definition 3. The quotient algebra $B(X)/K(X)$ whose elements are the cosets $T+K(X)$ is a Banach algebra under the quotient norm. (This is called the Calkin algebra.)

Definition 4. a) $l_\infty(X)$ is the linear space of bounded sequences $\{x_n\}$ of elements $x_n\in X$ with the supremum norm $\|\{x_n\}\|=\sup_n\|x_n\|$.

b) $m(X)$ is the linear subspace of $l_\infty(X)$ consisting of those sequences every subsequence of which contains a convergent subsequence.

It is elementary to check that $l_\infty(X)$ is a Banach space and $m(X)$ is closed. Further, for $T\in B(X)$,

$$\{x_n\}\in l_\infty(X)\Rightarrow\{Tx_n\}\in l_\infty(X)$$

$$\{x_n\}\in m(X)\Rightarrow\{Tx_n\}\in m(X).$$

Definition 5. a) $\hat{X}=l_\infty(X)/m(X)$

b) For $T\in B(X)$, $\hat{T}$ denote the operator on $\hat{X}$ defined by $\hat{T}(\{x_n\}+m(X))=\{Tx_n\}+m(X)$.

Clearly $\hat{T}\in B(\hat{X})$ and $T\in K(X)\Leftrightarrow\hat{T}=\hat{O}$.

II. A review of Atkinson's results of Fredholm operator

For $T\in B(X)$ the following statements are equivalent

(1) $T\in\Phi(X)$

(2) $T+F(X)\in\mathrm{Inv}(B(X)/F(X))$

(3) $T+K(X)\in\mathrm{Inv}(B(X)/K(X))$

(4) $\hat{T}\in\mathrm{Inv}(B(\hat{X}))$

Proof. (1)$\Rightarrow$(2) Assume $T\in\Phi(X)$, then $\alpha(T)<\infty$, TX is of finite codimension. Then there exist closed subspaces Z, W of X; $X=\mathrm{Ker}(T)\oplus Z=TX\oplus W$.

T can be drawn as the 2×2 matrix

$$T=\begin{pmatrix} O:\mathrm{Ker}(T)\to W & O:Z\to W \\ O:\mathrm{Ker}(T)\to TX & T_{22}:Z\to TX \end{pmatrix}$$

T_{22} is bijective and continuous and TX is closed. So there exists a continuous linear inverse S_{22} : $TX\to Z$ ([1] p. 57).

$$\text{If } S=\begin{pmatrix} O:W\to \mathrm{Ker}(T) & O:TX\to\mathrm{Ker}(T) \\ O:W\to Z & S_{22}:TX\to Z, \end{pmatrix}, \text{ then}$$

$$TS=\begin{pmatrix} O:W\to W & O:TX\to W \\ O:W\to TX & I:TX\to TX \end{pmatrix} \text{ and}$$

$$ST=\begin{pmatrix} O:\mathrm{Ker}(T)\to\mathrm{Ker}(T) & O:Z\to\mathrm{Ker}(T) \\ O:\mathrm{Ker}(T)\to Z & I:Z\to Z \end{pmatrix}$$

Clearly, TS and ST are projections of finite codimension

$\Rightarrow\exists$ projections $P,Q\in F(X)$: $TS=I-P$, $ST=I-Q$

$\Rightarrow S$ is the inverse of T modulo $F(X)$

i.e. $T+F(X)\in\mathrm{Inv}(B(X)/F(X))$

(2) $\Rightarrow$(3) is obvious

(3) $\Rightarrow$(4) $S+K(X)=(T+K(X))^{-1}$

$\Rightarrow EK_1, K_2\in K(X)$: $TS=I-K_1$, $ST=I-K_2$

$\Rightarrow\hat{S}\hat{T}=\hat{I}=\hat{T}\hat{S}$

$\Rightarrow\hat{T}\in\mathrm{Inv}(B(\hat{X}))$

(4) $\Rightarrow$(1) Let $\hat{T}\in\mathrm{Inv}(B(\hat{X}))$ and choose a sequence $\{x_n\}$ in the unit ball of $\mathrm{Ker}(T)$

i) $\{Tx_n\}=0\Rightarrow\hat{T}(\{x_n\}+m(X))=0$

$\Rightarrow\{x_n\}+m(X)=0$

$\Rightarrow\{x_n\}\in m(X)$

$\Rightarrow$the unit ball of $\mathrm{Ker}(T)$: compact

$\Rightarrow\alpha(T)<\infty$

ii) Next we show that TX is closed in X. Since $\alpha(T)<\infty$, there exists a closed subspace Z of X such that, $X=\mathrm{Ker}(T)\oplus Z$. Clearly $TX=T(\mathrm{Ker}(T)\oplus Z)=TZ$ and T is injective on Z, so it suffices to show that T is bounded below on Z.

Suppose not, there exist a $\{x_n\}\subset Z$: $\|x_n\|=1$ for each n and $Tx_n\to 0$.

$\{Tx_n\}\in m(X)\Rightarrow\hat{T}(\{x_n\}+m(X))=0$

$\Rightarrow\{x_n\}+m(X)=0$

$\Rightarrow\{x_n\}\in m(X)$

Thus there is a subsequence $\{x_{n_k}\}$: $x_{n_k}\to y\in X$

Then $\|y\|=1$ and $Tx_{n_k}\to Ty=0$, but $Z\cap \mathrm{Ker}(T)=(0)$ which is a contradiction.

$\therefore TX$ is closed.

iii) Since TX is a closed, X/TX is a Banach space, it remains to prove $\beta(T)<\infty$.

Let $\{y_n\}\subset X$ be a sequence satisfying $\|y_n+TX\|\leq 1$ for each n.

$\Rightarrow \exists \{x_n\}\subset X : \|y_n+Tx_n\|\leq 2$ for each n

$\hat{T}$: invertible

$\Rightarrow \exists \{w_n\}\in l_\infty(X) : \hat{T}(\{w_n\}+m(X))=\{y_n+Tx_n\}+m(X)$

$\Rightarrow \{T(w_n-x_n)-y_n\}\in m(X)$

$\Rightarrow \exists$ subsequence $\{w_{n_k}\}$, $\{x_{n_k}\}$, $\{y_{n_k}\}$: $T(w_{n_k}-x_{n_k})-y_{n_k}\to z\in X$

$\Rightarrow \|y_{n_k}+z+TX\|\to 0$ as $k\to\infty$ ($\because TX$ is closed)

$\Rightarrow \{y_n+TX\}$ has a convergent subsequence

$\Rightarrow$unit ball of X/TX is compact

$\Rightarrow \beta(T)<\infty$

i) ii) iii) $\Rightarrow T\in\Phi(X)$

III. Main theorems

If $T\in B(X)$, λ is said to be a Fredholm point of T if $\lambda-T\in\Phi(X)$ and the set of Fredholm points of T is denoted by $\Phi(T)$. If either $\alpha(T)<\infty$ or $\beta(T)<\infty$ we define the index $i(T)$ of T by $i(T)=\alpha(T)-\beta(T)$, and if $T\in B(X)$, $\rho(T)$, $\sigma(T)$ denote the resolvent set and spectrum of T, respectively.

I want to show that an index-zero Fredholm operator can be decomposed into the sum of an invertible operator plus a finite rank operator, and that for $T\in B(X)$ $\sigma(T)\setminus\{\lambda\in\Phi(T)$ and $i(\lambda-T)=0\}=\bigcap_{R\in K(X)}\sigma(T+R)$.

Theorem 1. $T\in\Phi(X)$ and $i(T)=0$

$$\Rightarrow \exists\ S\in F(X) : T+\lambda S\in \mathrm{Inv}(B(X)),\ \lambda\neq 0$$

Proof. As in the proof of II, we may write

$$T=\begin{pmatrix} O:\mathrm{Ker}(T)\to W & O:Z\to W \\ O:\mathrm{Ker}(T)\to TX & O:Z\to TX\end{pmatrix} \text{ where } \dim(W)=\dim(\mathrm{Ker}(T))$$

since $i(T)=0$.

Construct $S\in F(X)$ by means of the isomorphism

$$J:\mathrm{Ker}(T)\to W$$

$$S=\begin{pmatrix} J:\mathrm{Ker}(T)\to W & O:Z\to W \\ O:\mathrm{Ker}(T)\to TX & O:Z\to TX\end{pmatrix}$$

If $\lambda\neq 0$, $T+\lambda S\in\mathrm{Inv}(B(X))$

Theorem 2. $T\in B(X)$

$$\Rightarrow \sigma(T)\setminus\{\lambda\in\Phi(T) \text{ and } i(\lambda-T)=0\}=\bigcap_{R\in K(X)}\sigma(T+R)$$

Proof. The result may be restated as follows:

$$\{\lambda\in\Phi(T) \text{ and } i(\lambda-T)=0\}=\bigcup_{R\in K(X)}\rho(T+R)$$

($\supset$) Let $\lambda \in \bigcup_{R \in K(X)} \rho(T+R) \Rightarrow \lambda \in \rho(T+R_0)$ for some $R_0 \in K(X)$

$$\Rightarrow \lambda - T - R_0 \in \Phi$$
$$\Rightarrow i(\lambda - T - R_0) = 0$$
$$\Rightarrow \lambda - T \in \Phi(X),\ i(\lambda - T) = 0$$
$$\Rightarrow \lambda \in \Phi(T),\ i(\lambda - T) = 0$$

($\subset$) Let $\lambda - T \in \Phi(X)$, $i(\lambda - T) = 0$

Without loss of generality take $\lambda = 0$.

$\Rightarrow T \in \Phi(X)$, $i(T) = 0$

$\Rightarrow \exists\ R_1 \in K(X) : 0 \in \rho(T+R_1)$ by Theorem 1.

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