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Covering Structures on Semi-closure Spaces

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1. Introduction

In [3], the authors have first introduced and investigated the concept of semi-closure structures. The purpose of this paper is to state and prove the theorem which is similar to the classical theorem concerning with uniformity on topological spaces. In section 2, we shall introduce the concept of semi-open covering structures on a semiclosure space and show that some equivalent conditions to the s-continuities. In the final section, we shall give a main theorem; Let $\{A_i\}_{i\in I}$ be a semi-open covering structure on an s-compact space X and $\{B_i\}_{i\in I}$ a semi-open covering structure on a semi-closure space Y. If $f: X\rightarrow Y$ is s-continuous, then f is uniformly s-continuous.

Let X be a set and let $\mathcal{P}(X)$ be the power set of X. A mapping u: $\mathcal{P}(X)\rightarrow\mathcal{P}(X)$ is called a *semi-closure structure* on X if it satisfies i) $u(\phi)=\phi$, ii) $A\in u(A)$ for every $A\in\mathcal{P}(X)$ and iii) $u(A)=u\circ u(A)$ for every $A\in\mathcal{P}(X)$. If a semi-closure structure u on X is given, then (X,u) is called a *semi-closure space* and for a convenience, we shall agree to use u as $\{A|u(X-A)=X-A\}$. A set A is called a *semi-closed subset* of (X,u) if $u(A)=A$; it is *semi-open* if $X-A$ is semi-closed. Clearly, a semi-closure structure u is satisfied i) $X,\phi\in u$ and ii) for every $A_i\in u$, $i\in I$, $\bigcup_{i\in I} A_i\in u$, but the intersection of two elements of u is not an element of u, in general (A family u of subsets of X satisfying the above i) and ii) is called a *supratopology* [2] or *pretopology* [1] for X).

Let f be a mapping from a semi-closure space (X,u) into a semi-closure space (Y,w). If for every $A\in w$, $f^{-1}(A)\in u$ (resp. for every $A\in u$, $f(A)\in w$), then we shall say f is *s-continous* (resp. *s-open*).

2. Semi-open covering structures

Let $\mathscr{A}$ be a semi-open cover of a semi-closure space (X,u). We denote $A(x)$ an element of $\mathscr{A}$ such that $x\varepsilon A(x)$. Moreover, for each $x\varepsilon X$, let $S(x)=\bigcup_{A(x)\varepsilon\mathscr{A}} A(x)$. Let $\mathscr{A}_i$ and $\mathscr{A}_j$ be two semi-open covers of X. If for every $A_i\varepsilon\mathscr{A}_i$, there exists an $A_j\varepsilon\mathscr{A}_j$ such that $A_i\subset A_j$, then we shall write $i\leq j$. Based on the above notations, we introduce a semi-open covering structure on a semi-closure space (X,u). A family $\{\mathscr{A}_i\}_{i\varepsilon I}$ of semi-open covers of X is called a semi-open covering structure on X if i) for every semi-open set A of X with $X\varepsilon A$, there exists an $i\varepsilon I$ such that $\bigcup_{y\varepsilon S_i(x)} S_i(y)\subset A$, ii) for each $\mathscr{A}_i$ and $\mathscr{A}_j$ of $\{\mathscr{A}_i\}_{i\varepsilon I}$, $i\leq j$ or $j\leq i$.

Remark 2.1. Let us observe that $\leq$ is a reflexive and transitive relation in I. Moerover, every two elements of I are comparable with respect to $\leqslant$. However, $\leqslant$ need not be anti-symmetric.

Lemma 2.2. *The condition i) implies that for every semi-open set A of X with $x\varepsilon A$, there exists an $i\varepsilon I$ such that $S_i(x)\subset A$.*

This Lemma 2.2 will be used in the sequel.

Lemma 2.3. *Let $\{\mathscr{A}_i\}_{i\varepsilon I}$ be a semi-open covering structure on a semi-closure space (X,u). Let i and j be two elements of I such that $i\leq j$. If $A_i\cap S_j(x)\neq\phi$, then $A_i\subset \bigcup_{y\varepsilon S_j(x)} S_j(y)$.*

Proof. Since $i\leq j$, there exists an A_j such that $A_i\subset A_j$. Since $A_i\cap S_j(x)\neq\phi$, there exists $y\varepsilon A_i\cap S_j(x)$ and therefore $y\varepsilon A_i\subset S_j(y)$. Since $y\varepsilon S_j(x)$, $A_i\subset A_j\subset S_j(y)\subset \bigcup_{y\varepsilon S_j(x)} S_j(y)$. Thus $A_i\subset \bigcup_{y\varepsilon S_j(x)} S_j(y)$.

Theorem 2.4. *Let $\{\mathscr{A}_i\}_{i\varepsilon I}$ be a semi-open covering structure on a semi-closure space (X,u) and $\{\mathscr{B}_j\}_{j\varepsilon J}$ a semi-open covering structure on a semi-closure space (Y,w). Let f be a mapping from X to Y. Then f is s-continous if and only if for every $x\varepsilon X$ and every $B_j(f(x))$, there exists an $i\varepsilon I$ such that $f(S_i(x))\subset B_j(f(x))$.*

Proof. Assume that f is s-continuous. Let $f^{-1}(B_j(f(x))=V$. Since $B_j(f(x))$ is semi-open and f is s-continuous, V is a semi-open set of X such that $x\varepsilon V$. By Lemma 2.2, there exists an $i\varepsilon I$ such that $S_i(x)\subset V$ and therefore $f(S_i(x))\subset B_j(f(x))$.

Conversely, assume that for each $x\varepsilon X$ and every $B_j(f(x))$, there exists an $i\varepsilon I$ such that $f(S_i(x))\subset B_j(f(x))$. Let U be a semi-open set of Y such that $f(x)\varepsilon U$ with $x\varepsilon X$. By Lemma 2.2, there exists a $j\varepsilon J$ such that $S_j(f(x))\subset U$ and therefore, there exists a $B_j(f(x))$ such that $B_j(f(x))\subset S_j(f(x))$. Then by hypotheis, there exists an $i\varepsilon I$ such that $f(S_i(x))\subset B_j(f(x))\subset U$ Consequently, $S_i(x)\subset f^{-1}(U)$ and therefore, $f^{-1}(U)=\bigcup_{f(x)\varepsilon U} S_i(x)$ is a semi-open set of X implies that f is s-continuous.

Corollary 2.5. *Assume the hypotheis of Theorem 2.4. Then, f is s-continuous if and only if for every $x\varepsilon X$ and every $S_j(f(x))$, there exists an $i\varepsilon I$ such that$(f(S_i(x))\subset S_j(f(x))$.*

Theorem 2.4 and Corollary 2.5. can be stated as follows.

Theorem 2.6. *Assume the hypothesis of Theorem 2.4. Then, f is s-continuous if and only if for every $x\varepsilon X$ and every $j\varepsilon J$, there exists an $i\varepsilon I$ such that $f(S_i(x))\subset B_j(f(x))$.*

Corollary 2.7. *Assume the hypothesis of Theorem 2.4. Then, f is s-continuous if and only if for every $x\varepsilon X$ and every $j\varepsilon J$, there exists an $i\varepsilon I$ such that $f(S_i(x))\subset S_j(f(x))$.*

3. The main theorem

A semi-closure space (X,u) is said to be *s-compact* if every semi-open cover of X has a finite subcover. Let $\{\mathscr{A}_i\}_{i\varepsilon I}$ be a semi-open covering structure on a semi-closure space (X,u) and $\{\mathscr{B}_j\}_{j\varepsilon J}$ a semi-open covering structure on a semi-closure space (Y,w). Let f be a mapping from X into Y. The mapping f is called *uniformly s-continuous* if for every $j\varepsilon J$ there exists a $i\varepsilon I$ such that every $A_i\varepsilon\mathscr{A}_i$ is mapped by f into a $B_j\varepsilon\ \mathscr{B}_j$.

Remark 3.1. Every uniformly s-continuous mapping is s-continous, but the converse is not true, in general.

Theorem 3.2. *Let* $\{\mathscr{A}_i\}_{i\varepsilon I}$ *be a semi-open covering structure on an s-compact space X and* $\{\mathscr{B}_j\}_{j\varepsilon J}$ *a semi-open covering structure on a semi-closure space Y. If* $f: X \rightarrow Y$ *is s-continuous, then f is uniformly s-continuous.*

Proof. Let $j\varepsilon J$ be given. By Theorem 2.6, for every $x\varepsilon X$ there exists an $h\varepsilon I$ such that $f(S_h(x)) \subset B_j(f(x))$. Since $S_h(x)$ is semi-open and $\{\mathscr{A}_i\}_{i\varepsilon I}$ is a semi-open covering structure on X, there exists an $k\varepsilon I$ such that $\bigcup_{y\varepsilon S_k(x)} S_k(y) \subset S_h(x)$. Clearly, the set $\{S_k(x)|\cdots\}$ of all $S_k(x)$ given by as the above is a semi-open cover of X and since X is s-compact there exists a finite subfamily $\{S_n(x)|\cdots\}$ of $\{S_k(x)|\cdots\}$ which covers X. Let $F=\{S_n(x)|\cdots\}$ $\{S_k(x)|\cdots\}$. Then we obtain $\bigcup_{z\varepsilon S_n(x)} S_n(z) \subset S_k(x)$, for every $S_n(x)\varepsilon F$. By Remark 2.1, there exists an $i\varepsilon I$ such that $i\varepsilon n$ for every n with $S_n(x)\varepsilon F$. Let A_i be any element of A_i and let $y\varepsilon A_i$. Then there exists $S_n(x)\varepsilon F$ such that $y\varepsilon S_n(x)$, since F covers X. Therefore, $A_i \subset S_h(x)$ that is, $f(A_i) \subset f(S_h(x)) \subset B_j(f(x)) \varepsilon B_j$. Thus, for every $j\varepsilon J$, there exists an $i\varepsilon I$ such that every $A_i\varepsilon\mathscr{A}_i$, $f(A_i)\varepsilon\mathscr{B}_j$.

Corollary 3.3. *Assume the hypothesis of Theorem* 3.2. *Then, for every* $j\varepsilon J$ *there exists an* $i\varepsilon I$ *such that for every* $x\varepsilon X$, $S_i(x)$ *is mapped by f into* $S_j(f(x))$.

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