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On $\overline{FC}_A^-$ Groups

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1. Introduction

Let G be a locally compact group. We shall denote the automorphism group of G by $U(G)$ and the group of all inner automorphisms of G by $In(G)$. For an arbitrary subgroup A of $U(G)$, we denote the set of all elements $g \in G$ for which $\{f(g) \mid f \in A\}$ is precompact by $B_A(G)$. We recall that G is a $\overline{FC_A}^-$ group if $G = B_A(G)$.

Usually the case $A = In(G)$ is considered and in this case $\overline{FC}^-$ stands for $\overline{FC_A}^-$. In [1] the authors studied $B_A(G)$ for an arbitrary subgroup A of $U(G)$, and raised the following open problem;

Does every locally compact totally disconnected $\overline{FC}^-$ group have a compact invariant neighborhood of the identity?

This problem was actually solved by T.S.Wu and Y.K.Yu [2]. In this paper we shall generalize some of the results in the paper by modifying their techniques.

We shall need the following result [1. Theorem 3.11].

Theorem A *Let A be a subgroup of $U(G)$ containing $In(G)$, E be a precompact periodic A-invariant subset of G. Then the closed subgroup generated by E is a compact A-invariant subgroup.*

2. Locally compact groups with open $B_A(G)$

Theorem 1. *Let G be a locally compact group, Then $B_A(G)$ is open if and only if G has a compact A-invariant neighborhood of the identity.*

Proof. Suppose that G has compact A-invariant neighborhood N of the identity. Then clearly N is contained in $B_A(G)$. Since $B_A(G)$ is a subgroup of G, we conclude that $B_A(G)$ is open.

Conversely, suppose that $B_A(G)$ is open and G has no compact A-invariant neighborhoods of the identity. Let N be a compact neighborhood of the identity in G such that $N \subset B_A(G)$. Let V be an open symmetric neighborhood of the identity such that $\bar{V}^2 \subset N$. We shall construct sequences $\{x_n\}$ in G and $\{f_n\}$ in A satisfying the conditions

(1) $y_n = x_1 x_2 \ldots x_n \in V$ for all n.

(2) $f_n(x_{n+1} \ldots x_m) \in V$ for all n and m, $(m > n)$.

(3) N, $f_1(y_1)N, f_2(y_2)N, \ldots, f_k(y_k)N$ are disjoint for all k.

First we show that the existence of such sequences is sufficient for the proof. Since $y_n \in V \subset N$ for all n, there exists a subnet $\{y_{n_k}\}$ that converges to a point $y \in N$. Now, for each n

$$f_n(y)=\lim_k f_n(y_{n_k})=\lim_k f_n(y_n)\ f_n(x_{n+1}\dots x_{n_k})=f_n(y_n)\ \lim_k f_n(x_{n+1}\dots x_{n_k})$$

Hence $f_n(y)\in f_n(y_n)\bar{V}$ for each n. Suppose that $\{f_{n_k}(y)\}$ converges to some $x\in G$. Then $f_{n_k}(y)$ eventually lies in xV.

Hence $f_{n_k}(y)=xv_k$, $v_k\in V$ and we see that

$$x=f_{n_k}(y)v_k^{-1}\in f_{n_k}(y)\bar{V}\subset f_{n_k}(x_1\dots x_{n_k})\bar{V}^2\subset f_{n_k}(x_1\dots x_{n_k})N.$$

This contradicts the condition (3), and hence the net $\{f_{n_k}(y)\}$ does not converge. A similar argument shows that $\{f_{n_k}(y)\}$ has no convergent subnets. Therefore $\{f(y)\mid f\in A\}$ is not precompact. But $y\in N\subset B_A(G)$. This is a contradiction.

Now let us start the construction. There exist $x_1\in V$ and $f_1\in A$ such that $N\cap f_1(x_1)N=\phi$. Otherwise, suppose N meets each set $f(x)N$, where $x\in V$, $f\in A$. Then $f(V)\subset NN^{-1}$. Thus, if we let $A_0=\bigcup_{f\in A}f(V)\subset NN^{-1}$, we have a compact A-invariant neighborhood $\bar{A}_0$ of the identity; this is a contradiction to our initial assumption.

Suppose that we have constructed the sets $\{x_1,\dots,x_k\}$ and $\{f_1,\dots,f_k\}$ satisfying the conditions (1), (2), (3) for all $n\leqq k$. Now, consider the continuous mappings

$$x\longmapsto y_kx$$
$$x\longmapsto f_i(x_{i+1}\dots x_kx)\quad(i=1,\dots,k-1)$$
$$x\longmapsto f_k(x)$$

Since each of the mappings maps e into V, some neighborhood W of the identity is mapped into V by these mappings. Let

$$R_k=N\cup f_1(y_1)N\cup f_2(y_2)N\ \dots\cup f_k(y_k)N$$

Suppose $R_k\cap f(y_kx)N\neq\phi$ for all $x\in W$ and $f\in A$. Then

$$A_k=\bigcup_{f\in A}f(y_kW)\subset R_kR_k^{-1}$$

and $\bar{A}_k\bar{A}_k^{-1}$ is a compact A-invariant neighborhood of the identity. This is a contradiction. Hence, there exist $x_{k+1}\in W$ and $f_{k+1}\in A$ such that $R_k\cap f_{k+1}(y_kx_{k+1})N=\phi$. The construction is complete. In the following two corollaries, we assume that $A\supset In(G)$.

Corollary 1. *If G is a locally compact, totally disconnected group, then $B^A(G)$ is open if and only if G has a compact open A-invariant subgroup.*

Proof. Suppose $B^A(G)$ is open; then there exists a compact A-invariant neighborhood N of the identity. Let K be a compact open subgroup of G contained in N. Then $\bigcup_{f\in A}f(K)$ is precompact A-invariant, and periodic. Hence, by Theorem A, it is contained in a compact A-invariant open subgroup of G. The converse is clear.

Corollary 2. *Let G be a locally compact totally disconnected $\overline{FC}_A$ group and let K be a compact subgroup of G. Then K is contained in a compact open A-invariant subgroup of G.*

Proof. Let $k\in K$. Then $\{f(k)\mid f\in A\}$ is precompact, A-invariant and periodic. Hence, by Theorem A, it is contained in a compact, A-invariant subgroup N_k of G. By Corollary 1, there exists a compact open A-invariant subgroup N of G. Clearly, K is covered by finitely many compact open A-invariant subgroups of the form NN_k. Their union is compact, periodic and A-invariant, hence, again by Theorem A, it is contained in a compact A-invariant subgroup of G. This completes the proof.

References

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