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A Note on the Semi-Continuity in Topological Space.

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Abstract

In this paper, we investigate the properties of the semi-continuous functions on the first axiom space, n-th product space, pseudo-metric space, and proximity space. Counterexample is used when the converse of the theorem is not hold.

Introduction.

Definition 1. A set A in a topological space X will be termed ***semi-open*** (will be written ***s.o.***) if and only if there exists an open set O such that $O\subset A\subset cl(O)$ where $cl(O)$ denotes the closure operator of O.

Definition 2. Let $f: X\to X^*$ be single valued (continuity is not assumed) where X and X^* are topological spaces. Then $f: X\to X^*$ is termed ***semi-continuous*** (will be written ***s.c.***) if and only if, for open O^* in X^*, $f^{-1}(O^*)\in S.O.(X)$, where $S.O.(X)$ denotes the class of all semi-open sets in X.

Theorems

Theorem 1. *Let $f: X\to X^*$ be continuous where X and X^* are topological spaces. Then f is semi-continuous.*

Example 1. Converse of Thm. 1 is not hold.

Let $X=X^*=[0,1]$ and $f: X\to X^*$ as follows,

$$f(x)=\begin{cases}1, & if\ \ 0\leq x\leq \frac{1}{2}\\ 0, & if\ \ \frac{1}{2}<x\leq 1\end{cases}$$

Then f is semi-continuous but not continuous.

Theorem 2. *Let $f: X\to X^*$ be a single valued function, X and X^* are topological spaces. Then $f: X\to X^*$ is s.c. iff for $f(p)\in O^*$, there exists an $A\in S.O.(X)$ such that $p\in A$ and $f(A)\subset O^*$.*

Proof. Let O^* be open in X and $p\in f^{-1}(O^*)$. Then $f(p)\in O^*$ and thus there exists an $A_p\in S.O.(X)$ such that $p\in A_p$ and $f(A_p)\subset O^*$, Then $p\in A_p\subset f^{-1}(O^*)$ and $f^{-1}(O^*)=\bigcup_{p\in(f^{-1}O^*)} A_p$. For some O_p, $O_p\subset A_p\subset cl(O_p)$ since $A_p\in S.O.(X)$. Then $\bigcup_p O_p\subset \bigcup_p A_p\subset \bigcup_p cl(O_p)\subset cl(\bigcup_p O_p)$. Hence let $O=\bigcup_p O_p$, $f^{-1}(O^*)\in S.O.(X)$.

On the other other hand, let $f(p)\in O^*$. Then $p\in f^{-1}(O^*)\in S.O.(X)$, since $f: X\to X^*$ is s.c. Let $A=f^{-1}(O^*)$. Then $p\in A$ and $f(A)\subset O^*$.

Thorem 3. *Let $f: X \to X^*$ be s.c. and X^* a first axiom space. Let P be the set of points of discontinuity of f. Then P is of first category.*

Proof. Let $p \in P$. Then there exists an $O_{i_p}^*$ in the countable open basis for X^* such that $p \in O$ open in X implies that $f(O) \not\subset O_{i_p}^*$. Then there exists an $A_{i_p} \in S.O.(X)$ such that $p \in A_{i_p}$ and $f(A_{i_p}) \subset O_{i_p}^*$ by Thm. 2. But $A_{i_p} = O_{i_p} \cup B_{i_p}$ where $B_{i_p} \subset cl(O_{i_p}) - O_{i_p}$ is nowhere dence set. Hence $p \notin O_{i_p}$ and thus $p \in B_{i_p}$. It follows $P \subset \bigcup_{p \in P} B_{i_p}$ and since $\bigcup_{p \in P} B_{i_p}$ is of first category, P is of first category.

Example 2. Converse of Thm. 3 is not hold generally.

Let $X = [0, 1]$ and $X^* = [0, 1]$. Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1/q, & \text{if } x \text{ is rational } (x = p/q,\ p \text{ and } q \text{ are relativel prime}) \end{cases}$$

Then f is continuous at the irrationals and discontinuous at the rationals. Hence the sets are of the first category. But $f: X \to X^*$ is not s.c. since $f\left(\frac{1}{2}, 1\right]$ is a subset of the rationals and thus is not s.o.

Theorem 4. *Let $f_i: X_i \to X_i^*$ be s.c. for $i = 1, 2, \cdots, n$. Let $f: \prod_{i=1}^n X_i \to \prod_{i=1}^n X_i^*$ as follows; $f(x_1, x_2, \cdots, x_n) = (f(x_1), f_2(x_2), \cdots, f_n(x_n))$. Then $f: \prod_{i=1}^n X_i \to \prod_{i=1}^n X_i^*$ is s.c.*

Proof. Let $\prod_{i=1}^n O_i^* \subset \prod_{i=1}^n X_i^*$ where O_i^* is open in X_i^* for $i = 1, 2, n$. Then $f^{-1}(\prod_{i=1}^n O_i^*) = \prod_{i=1}^n f^{-1}(O_i^*)$. But $f_i^{-1}(O_i^*)$ are s.o. in X_i for $i = 1, 2, \cdots, n$, and thus $\prod_{i=1}^n (O_i^*)$ is s.o. in $\prod_{i=1}^n X_i$. Now if O^* is any open set in $\prod_{i=1}^n X_i^*$, then $f^{-1}(O^*) = f^{-1}(\bigcup O_\alpha^*)$ where O_α^* is of the form $\prod_{i=1}^n O_{\alpha i}^*$. Then $f^{-1}(O^*) = \bigcup f^{-1}(O_\alpha^*)$ which is s.o. since $f^{-1}(O_\alpha^*)$ is s.o. by the above argument.

Theorem 5. *Let $h: X \to \prod_{i=1}^n X_i$ be s.c. where X and X_i for $= 1, 2, \cdots n$ are topological spaces. Let $f_i: X \to X_i$ as follows; $h(x) = (x_1, x_2, \cdots x_n)$ for $x \in X$ and $x_i \in X_i$. Let $f_i(x) = x$. Then $f_i: X \to X_i$ is s.c. for $i = 1, \cdots 2, n$.*

Proof. It is enough to show only $f_1: X \to X_1$ is s.c. Let O_1 be open in X_1. $O_1 \times \prod_{i=2}^n X_i$ is open in $\prod_{i=1}^n X_i$ and $h^{-1}(0_1 \times \prod_{i=2}^n X_i)$ is s.o. in X. But $f_1^{-1}(0_1) = h^{-1}(0_1 \times \prod_{i=2}^n X_i)$ and thus $f_1: X \to X_1$ is s.c.

Example 3. Converse of Thm. 5 is not hold generally. Let $h: X \to X_1 \times X_2$ and $f_i: X \to X_i$ for $i = 1, 2$. Let $X = X_1 = X_2 = [0, 1]$. Let $f_1: X \to X_1$ and $f_2: X \to X_2$ as follows;

$$f_1(x) = \begin{cases} 1, & \text{if } 0 \leqq x \leqq \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} < x \leqq 1 \end{cases}$$

$$f_2(x) = \begin{cases} 1, & \text{if } 0 \leqq x < \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

Then f_1 and f_2 are s.c., but

$h(x) = (f_1(x), f_2(x)): X \to X_1 \times X_2$ is not s.c. for $S_{1/2}(1, 0)$ is open in $X_1 \times X_2$, but $h^{-}(S_{1/2}(1, 0)) = \left(\frac{1}{2}\right)$ which is not s.o. in X. $S_{1/2}(1, 0)$ denotes the spherical neighborhood of $(1, 0)$, radius $\frac{1}{2}$.

Remark. A s.c. function of a s.c. function is not s.c. in general, in fact,

Example 4. Let $X=X_1=X_2=[0,1]$. Let $f_1 : X\to X_1$ and $f_2 : X\to X_2$ as follows;

$$f_1(x)=\begin{cases} x, & \text{if } 0\leqq x\leqq \frac{1}{2} \\ 0, & \text{if } \frac{1}{2}<x\leqq 1 \end{cases} \qquad f_2(x)=\begin{cases} 0, & \text{if } 0\leqq x<\frac{1}{2} \\ 1, & \text{if } \frac{1}{2}\leqq x\leqq 1 \end{cases}$$

Then f_1 and f_2 are s.c. but $f_2\cdot f_1$ is not s.c.

Example 5. Let $X=X^*=[0,1]$ and $f_n : X\to X^*$ be defined as $f_n(x)=x^n$ for $n=1,2,\cdots$ Then f_0: $X\to X$ is the limit of the sequence where

$$f_0(x)=\begin{cases} 0, & \text{if } 0\leqq x<1 \\ 1, & \text{if } x=1. \end{cases}$$

But f_0 is not s.c. For $\left(\frac{1}{2},1\right]$ is open in X^*, but $f_0\left(\frac{1}{2},1\right]=(1)$ which is not s.o. in X. Thus the limit of a sequence of s.c. functions is not s.c. in general.

Theorem 6. *Let $f_n : P\to P^*$, where P and P^* are pseudo-metric spaces with pseudo-metrics d and d^*, be s.c. for $n=1,\cdots 2$, and let $f_0 : P\to P^*$ be the uniform limit of $\{f_n\}$. Then $f_0 : P\to P^*$ is s.c.*

Proof. Let $f_0(x)\in O^*$. Then $f_0(x)\in S_\eta^*(f_0(x))\subset O^*$ for some $\eta>0$. There exists then an n^* such that $d^*(f_n(y), f_0(y))<\frac{\eta}{2}$ for all $y\in P$. Then $d^*(f_{n^*}(x), f_0(x))<\frac{\eta}{2}$ and thus $(f_{n^*}(x)\in S_{\eta/2}^*(f_0(x))\subset O^*$. Since f_{n^*} is s.c. there exists, by Thm 2, an A s.o. such that $x\in A$ and $f_{n^*}(A)\subset S_{\eta/2}^*(f_0(x))$. The proof will be complete when we show that $f_0(A)\subset O^*$. Let $y\in A$. Then $d^*(f_0(y), f_0(x))\leqq d^*(f_0(y), f_{n^*}(y))+d^*(f_{n^*}(y), f_0(x))<\frac{\eta}{2}+\frac{\eta}{2}=\eta$. This shows $f_0(A)\subset S_\eta^*(f_0(x))\subset O^*$.

Theorem 7. *A proximity mapping $f : (X,\delta_1)\to(Y,\delta_2)$ is s.c. with respect to $\mathcal{T}(\delta_1)$ and $\mathcal{T}(\delta_2)$.*

Proof. Note that a function $f : (X,\delta_1)\to(Y,\delta_2)$ is proximity mapping iff $A\delta_1 B$ implies $f(A)\delta_2 f(B)$, where (X,δ_1) and (Y,δ_2) are two proximity spaces. $\mathcal{T}(\delta_1)$ and $\mathcal{T}(\delta_2)$ are topologies on X *and* Y respectively which are induced by a proximity on X and Y respectively. This result follows easily from the fact that $x\delta_1 A$ implies $f(x)\delta_2 f(A)$. Therefore, f is continuous with respect to $\mathcal{T}(\delta_1)$. Hence f is semicontinuous with respect to $\mathcal{T}(\delta_1)$ and $\mathcal{T}(\delta_2)$.

요 약

이 논문에서는 Norman Levin의 논문에서 나타난 semi-open의 개념을 사용하여 정의된 semi-contin uous의 여러가지 성질을 조사하였고, semi-continuity를 위상공간의 first-axiom공간 pseudo-metric공간과 n-th product공간까지 조사하였으며, semi-continuous함수의 합성과 함수열의 극한과 proximity공간의 mapping에 대하여 조사하였다. 그리고 정리의 역이 성립하지 않는 경우 반례를 들어 보였다.

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