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A Note on Almost Primitive Rings

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1. Introduction

Let R be a ring and let M be a left R-module. M is called a faithful R-module if, for any $a\in R$, $ax=0$ for all $x\in X$ implies $a=0$ and M is irreducible provided $RM\neq 0$ and M has no proper submodules. Let $S=Hom_R(M, M)$. Then M becomes an right S-module with a module multiplication $mf=f(m)$. If M is an irreducible R-module, then S is a division ring and so the right S-module M_S is a vector space over S.

In this paper we give some definitions to generalize a well known theorem. That is, we define that a left R-module M over a ring R is almost irreducible if, for any $m\in M$, $Rm=0$ implies $m=0$ and for any proper R-submodule N of M, $Nf=0$ for some nonzero $f\in S=Hom_R(M, M)$, while also M_S is a free S-module. We say that a ring R is almost primitive if there is a faithful almost irreducible R-module M, which is called a characteristic R-module. In fact, almost primitive rings are discussed and some useful results are obtained in [1]. In section 3, we will show that there are sufficiently many examples of almost primitive rings which are not primitive and thus the definition of almost primitive ring with our main THEOREM is meaningful.

There is a well known theorem [cf. 5]: Let V be a nonzero vector space over a division ring D and let R be the endomorphism ring $Hom_D(V, V)$. If $g: V\to V$ is a homomorphism of additive groups such that $gr=rg$ for all $r\in R$, then there exists $d\in D$ such that $g(v)=dv$ for all $v\in V$.

We will generalize this theorem as follows.

THEOREM. *Let R be an almost primitive ring with a characteristic module ${}_RM\neq 0$ and let $S=Hom_R(M, M)$. Then M_s is a free S-module. Let $D=Hom_S(M_S, M_S)$. If $g: M\to M$ is a homomorphism of additive groups such that $gd=dg$ for all $d\in D$, then there exists $s\in S$ such that $g(m)=ms$ for all $m\in M$.*

2. Proof of THEOREM

For each $r\in R$ the map $\alpha_r: M\to M$ given by $\alpha_r(m)=rm$ is easily seen to be S-endomorphism of M, that is, $\alpha_r\in D=Hom_S(M_S, M_S)$. Furthermore, for all $r, s\in R$,

$$\alpha_{(r+s)}=\alpha_r+\alpha_s \text{ and } \alpha_{rs}=\alpha_r\alpha_s.$$

Consequently the map $\alpha: R\to D$ defined by $\alpha(r)=\alpha_r$ is a well defined homomorphism of rings. Since M is a faithful R-module, $\alpha_r=0$ if and only if $r=0$. Therefore α is a monomorphism, whence R is isomorphic to the subring $Im\alpha$ of $D=Hom_S(M_S, M_S)$. In this sence we identify R as a subring of D.

Let $\{u\}$ be a linearly independent subset of M over S. We claim that u and $g(u)$ are linearly dependent over S. To prove this, suppose that u and $g(u)$ are linearly independent over S. We will show a contradiction by a series of propositions.

(2. 1) *$Ru=M$ and $Rg(u)=M$.*

Proof. Since Ru is a R-submodule of the characteristic R-module M, $Ru\neq M$ implies $R(uf)=(Ru)f=0$ for some nonzero $f\in S$. Since M is almost irreducible $R(uf)=0$ implies $uf=0$, contradicting the fact that $\{u\}$ is linearly independent over S. Thus we have $Ru=M$. By similar argument, we also have $Rg(u)=M$.

(2. 2) *Let $W=g(u)S$ and let $A(W)=\{r\in R : rW=0\}$. Then, for any $a\in M$, $A(W)a=0$ implies $a\in W$.*

Proof. If $x\in R$ and $xg(u)=0$, then $x\in A(W)$, so that $xa=0$. Thus there is a well-defined R-endomorphism θ from $Rg(u)=M$ to $Ra\subseteq M$ given by $\theta : xg(u)\rightarrow xa$ for all $x\in R$. Since $R(g(u)\theta-a)=0$, we have $g(u)\theta-a=0$, so that $g(u)\theta=a\in W$.

(2. 3) *$A(W)u=M$.*

Proof. Suppose $A(W)u\neq M$. Since $A(W)u$ is a proper R-submodule of M, there exists a nonzero element f of S such that $A(W)(uf)=(A(W)u)f=0$. Then, by (2. 2), we have $uf\in W$. Since $f\neq 0$, this contradicts the fact that u and $g(u)$ are linearly independent over S. Therefore, $A(W)u=M$.

(2. 4) *There is a contradiction and this comes from assuming that u and $g(u)$ are linearly independent over S. Thus u and $g(u)$ are linearly dependent over S.*

Proof. Since $A(W)u=M$, $g(A(W)u)=g(M)$. Note that $gd=dg$ for all $d\in D$ and $A(W)\subseteq R\subseteq D$. Thus we have

$$g(M)=g(A(W)u)=A(W)g(u)=0.$$

That is, $g(u)\in g(M)=0$, contradicting the fact that u and $g(u)$ are linearly independent over S.

We have proved our claim, that is, u and $g(u)$ are linearly dependent over S. Thus we have $g(u)=us$ for some $s\in S$. Note that $Ru=M$ (by (2. 1)) and $R\subseteq D$. If $m\in M$ then there exists $d\in R\subseteq D$ such that $d(u)=du=m$. Therefore we have

$$g(m)=g(du)=gd(u)=dg(u)=d(us)=(d(u))s=ms.$$

This completes the proof.

3. Examples

We now show that there are sufficiently many examples of almost primitive rings which are not primitive. In fact, these examples show that there are many examples of almost primitive rings which are not primitive and are commutative. We need the following two lemmas [2].

Lemma 1. *Let KG be a group ring of a group G over a field K. Then KG is prime if and* *only if G has no nonidentity finite normal subgroup.*

Lemma 2. *Let G be a locally finite countable group. Then KG is primitive if and only if it is prime and semisimple.*

Let $G \neq \{1\}$ be a finite group and let K be a field with char $K \nmid |G|$. Then G has a nonidentity finite normal subgroup G itself. Thus, by Lemma 1, G is not prime and so KG is not prime and so KG is not primitive by Lemma 2.

To show that KG is almost primitive, consider the regular KG-module ${}_{KG}KG$. Since $1 \in KG$, it is easily shown that ${}_{KG}KG$ is a faithful KG-module. Moreover, $S = Hom_{KG}(KG, KG) \cong KG$ as rings. Since $1 \in KG$, it is clear that KG_{KG} is a free KG-module, that is, KG_S is a free S-module. Note that if char $K \nmid |G|$, the group ring KG is semisimple by Maschke's theorem. Thus every KG-module is completly reducible. Therefore, for any proper KG-submodule ${}_{KG}N$ of ${}_{KG}KG$, $KG = N \oplus N'$ for some KG-submodule ${}_{KG}N'$ of ${}_{KG}KG$. Then the projection $\pi_{N'}$ on N' is a nonzero element of $S = Hom_{KG}(KG, KG)$ and $N\pi_N = 0$. Finally it is clear that for any $x \in KG$, $KGx = 0$ implies $x = 0$. We have proved that the group ring KG is almost primitive.

Remark. There are many finite abelian group. Thus, for finite abelian group G and for a field K with char $K \nmid |G|$, the group ring KG is commutative, almost primitive but not primitive.

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