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Limiting Law for the Extreme Values of Variables in $G/M/1$ System

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ABSTRACT

Limiting law for maximum queue length in $G/M/1$ queue system is derived. Furthermore, a simple approach using mixing sequence is also discussed.

1. Introduction

Gnedenko's (1943) fundamental limit theorems for extreme values on the independent identically distributed random variables have been extended to dependent processes by Berman (1962, 1964). The limiting distribution usually becomes degenerate which is not of great interest in most cases. Thus we derive the limiting law for the normalized random variables in nondegenerate cases. Loynes (1965) and O'Brien (1974) have shown that nondegenerate extreme value of mixing sequences in the domain of attraction of

$$1)\ \Phi_1(x)=\begin{cases}\exp(-(-x)^{\alpha}), & x<0,\ \alpha>0,\\ 1, & x\geq 0,\end{cases} \tag{1-1}$$

$$2)\ \Phi_2(x)=\begin{cases}0, & \\ \exp[-x^{-\alpha}], & x>0,\ \alpha>0,\end{cases} \tag{1-2}$$

$$3)\ \Phi_3(x)=\exp[-e^{-x}], \qquad -\infty<x<\infty. \tag{1-3}$$

In this paper we study the maximum queue length in $G/M/1$ system where arrival distribution is $A(t)$ and service distribution function is $1-e^{-t/b}$, $b>0$. Let Y_k represent the number of customers present in the system just prior to the kth arrival. Our aim is to find the nondegenerate limiting law for $Q_n=\max_{1\leq k\leq n} Y_k$.

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2. Extremes with Random Sample Size

Let $X_1, \cdots, X_n$ be i.i.d. random variables with common distribution $F(\cdot)$ Let R_n be a positive integer valued random variable which is dependent on the sample size n. Then Galambos (1978) has shown the following lemma.

Lemma 2.1. Let $X_1, \cdots, X_n$ be i.i.d. random variable, $Z_n = \max_{1 \le k \le n} X_k$ and $Z_{R_n} = \max_{1 \le k \le R_n} X_k$. Assume that $R_n/n \to t$ in probability where t is a positive random variable and

$$\lim_{n \to \infty} P(Z_n \le a_n + b_n x) = H(x).$$

Then,

$$\lim_{n \to \infty} P(Z_{R_n} \le a_n + b_n x) = \int H^t(x) dP(t < x) \tag{2-1}$$

where $H^t(x) = H(A_t + B_t x)$ for some constants A_t and B_t.

If t is a degenerate random variable, then $\int H^t(x) dP(t < x) = H^t(x)$. Therefore non-degenerate limiting distributions of Z_n and Z_{R_n} are of the same type which implies that Z_n is in the same domain of attraction of Z_{R_n} (See Ash (1972)).

3. Limiting Law for Q_n

If we assume that the system is stable, then busy period and idle period form alternating renewal process. Let R_n be the number of busy periods completed just prior to the nth arrival and $U_i + 1$ be the maximum queue length in the ith busy period. Then we have

$$\max_{1 \le k \le R_n} U_k \le Q_n \le \max_{1 \le k \le R_n + 1} U_k \tag{3-1}$$

where U_i are independent and identically distributed random variables. Cohen (1967) has found the exact distribution of U_i and also shown that $P(U_1 \ge h) \sim a\lambda^h$ for large h where $a = \int x dA(x)$ and λ is a root of $z - \alpha\{b^{-1}(1-z)\}$, $\alpha(\theta) = \int e^{-\theta x} dA(x)$. Furthermore, Heyde (1971) has derived one discrete limiting form of Q_n by calculating the probability of $\max_{1 \le k \le R_n} U_k$. Here we derive the general form by giving the domain of attraction and Heyde's result in a simple way.

Lemma 3.1. $\dfrac{R_n}{n} \to \dfrac{1}{\mu}$ in probability where μ is the mean value of the number of customers served in a busy period.

Proof. See Heyde (1971).

Lemma 3.2. Let $Z_n = \max_{1 \le k \le n} U_k$ and $Z_{R_n} = \max_{1 \le k \le R_n} U_k$. Then Z_n and Z_{R_n} are in the same domain of attraction.

Proof. The proof follows from Lemma 2.1 and Lemma 3.1.

Theorem 3.3. Q_n is in the domain of attraction of $\Phi_3(x)$ for nondegenerate case.

Proof I. For any constant c we have

$$P(Z_n\leq x+c\ \ln(n))=P\left\{\bigcap_{i=1}^{n}(U_i\leq x+c\ \ln(n)\right\}$$
$$=\{1-P(U_i>x+c\ln(n))\}^n$$
$$=\exp\{n\ \ln[1-P(U_i>x+c\ \ln(n)]\}. \tag{3-2}$$

Since $P(U_i>x)\sim a\lambda^x$ for large x, we have

$$P(Z_n\leq x+c\ \ln(n))=P\left[\bigcap_{i=1}^{n}(U_i\leq x\ +c\ln(n))\right]$$
$$=[1-P(U_i>x+c\ \ln(n))]^n$$
$$\sim\exp[-an\lambda^{x+c\ln(n)}]\ \text{for large } n.$$

Let $W_n=\dfrac{Z_n-c_1-c\ \ln(n)}{c_2}$ where $c_2=\ln\left(\dfrac{1}{\lambda}\right)$, $c_1=\dfrac{\ln a}{c_2}$ and $c=-\dfrac{1}{\ln\lambda}$.

Thus $P(W_n\leq x)=P(Z_n\leq c_1+c_2x+c\ \ln(n))$

$$\sim\exp[-a\lambda^{c_1+c_2x}]$$
$$=\exp[-e^{c_1c_2}e^{-c_2(c_1+c_2x)}]=\exp[-e^{-x}].$$

Then W_n is of the same positive type as Z_n and W_n is in the domain of attraction of $\Phi_3(x)$. Therefore the proof follows from the inequality (3-1) and Lemma 3-2.

Proof II. Instead of calculating distribution of Z_n we use the sufficient condition of extreme value theory.

Let $V_i=c_2U_i$. Then V_i and U_i are of the same positive type. Thus it is sufficient to show that $L_n=\max\limits_{1\leq k\leq n} V_k$ is in the domain of attraction of $\Phi_3(x)$. Since

$$\frac{P(V_i>t+x)}{P(V_i>t)}=\frac{P(U_i>\frac{1}{c_2}(t+x))}{P(U_i>\frac{1}{c_2}t)}$$
$$\sim\frac{a\ \lambda^{\frac{1}{c_2}(t+x)}}{a\ \lambda^{\frac{1}{c_2}t}}=\lambda^{\frac{x}{c_2}}=e^{-x},$$

the proof follows. (See Galambos (1978)).

The theorem gives a general type of limiting law. The next theorem gives the detailed information on the limit behavior of Q_n.

Theorem 3.4. $\lim\limits_{n\to\infty}\{P(Q_n\leq x+c\ \ln(n))-\exp[-a\lambda^{x+c(\ln(n)-\ln(\mu))}]\}=0.$

Proof. Equation (2-1) and (3-2) give

$$P(Z_{R_n}\leq x+c\ln(n))\sim\exp[-a\lambda^{(A+B_x)+c\ln(n)}]$$

for large n, where $B=1$ and $A=c\ \ln(\mu)$. (See Galambos (1978)).

Since the inequality (3-1) gives

$$P(\max_{1\le k\le R_n+1} U_k \le x+c\ \ln(n)) \le P(Q_n \le x+c\ \ln(n))$$

$$\le P(\max_{1\le k\le R_n} U_k \le x+c\ \ln(n)),$$ the proof follows.

Remark. The continuous version of Heyde's result follows if we choose $c=1/c_2$.

4. Extreme Values in Mixing Sequences

The previous results can be approached through the extreme value distributions in strong mixing sequence of random variables. Loynes (1965) has shown that the possible limit forms for the maxima of φ-mixing sequences are the same as what we have in i.i.d. cases. Since $\{Y_k : k\ge 1\}$ forms a φ-mixing, the limiting law of Q_n is one of the three types. (See Billingsley (1968)). Furthermore, any real valued function of φ-mixing is also φ-mixing under Doeblin's condition (See Doob (1953)). Therefore possible nondegenerate limit law of waiting time is of the same type.

5. Comments

The domain of attraction problems using mixing sequence also have been studied by Denzel and O'Brien (1975). However explicit form as the results given in Section III is very hard to derive. Thus it would be a nice subject to study and also the limiting law of extreme values of waiting time can be derived similarly.

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