⟨Research Paper⟩

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On the Largest Optimal Stopping Time.

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ABSTRACT

The structure of the largest optimal stopping time in the discrete parameter processes is obtained by using the Doob decomposition of supermartingales.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. We denote $N=\{0,1,2,\cdots\}$ and $\bar{N}=N\cup\{+\infty\}$. Let $(\mathscr{F}_n)_{n\in\bar{N}}$ be an increasing sequence of sub-σ-fields of $\mathscr{F}$, and $(X_n)_{n\in\bar{N}}$ a sequence of integrable random variables such that X_n is $\mathscr{F}_n$-measurable. A function $T:\Omega\to\bar{N}$ is a stopping time if $\{T\leq n\}\in\mathscr{F}_n$ for all $n\in\bar{N}$. For a stopping time T, a new random variable X_T is defined by $X_T(\omega)=X_{T(\omega)}(\omega)$. An optimal stopping time is a stopping time T_0 such that $E(X_{T_0})=\sup_T E(X_T)$, where supremum is taken over the set of all stopping times.

Snell(1953) studied the existence and structure of the smallest optimal stopping time for discrete parameter stochastic processes. Later, the general theory of optimal stopping times was developed by Chow, Robbins, and Siegmund(1971). In this paper we study the structure of the largest optimal stopping time for discrete parameter stochastic processes. The largest optimal stopping time T means the optimal stopping time T and $S\leq T$ for any optimal stopping time S.

$(X_n)_{n\in\bar{N}}$ is a **supermartingale** if X_n^- is integrable and $E(X_m|\mathscr{F}_n)\leq X_n$ for each $n\leq m$. Note that if $(X_n)_{n\in\bar{N}}$ is a supermartingale, then by the optional sampling theorem $E(X_T|\mathscr{F}_S)\leq X_S$ holds for any stopping times S and T with $S\leq T$.

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The following theorem is the fundamental one to study optimal stopping times.

Theorem 1. Let $(X_n)_{n\in N}$ be a sequence of integrable random variables. Then there is a minimal supermartingale $(Z_n)_{n\in N}$ satisfying $X_n \leq Z_n$ for all n. Furthermore

$$Z_n = \operatorname{ess}\sup_{S\geq n} E(X_S|\mathscr{F}_n).$$

$$Z_n = \max\{X_n,\ E(Z_{n+1}|\mathscr{F}_n)\},$$

$$E(Z_n) = \sup_{S\geq n} E(X_S),$$

$$Z_\infty = X_\infty.$$

In this case, $(Z_n)_{n\in N}$ is called the **Snell Envelope** of given $(X_n)_{n\in N}$.

Proof. The proof of this theorem is similar to the proof in the case $(X_n)_{n\in N}$ with $E(\sup X_n^+) < +\infty$ (see Neveu(1975) Proposition VI-1-2).

Lemma 2. Let $(Z_n)_{n\in N}$ be the Snell Envelope of $(X_n)_{n\in N}$. A stopping time T is an optimal stopping time if and only if $X_T = Z_T$ and $(Z_{T\wedge n})_{n\in N}$ is a martingale.

Proof. By optional sampling theorem for the supermartingale $(Z_n)_{n\in N}$, we have $E(Z_T|\mathscr{F}_0) \leq Z_0$ and $E(Z_{T\wedge n}|\mathscr{F}_{n-1}) \leq Z_{T\wedge(n-1)}$.
Thus $(Z_{T\wedge n})_{n\in N}$ is a martingale i.e., $Z_{T\wedge n} = E(Z_T|\mathscr{F}_n)$ if and only if $E(Z_T) = E(Z_0)$.
On the other hand, we have, for every stopping time T,

$$X_T \leq Z_T,$$

$$E(X_T) \leq E(Z_0) = \sup_S E(X_S).$$

Thus we obtain that

T is optimal stopping time iff $E(X_T) = E(Z_0)$

iff $E(X_T) = E(Z_T) = E(Z_0)$

iff $X_T = Z_T,\ E(Z_T) = E(Z_0)$

iff $X_T = Z_T,\ (Z_{T\wedge n})_{n\in N}$ is a martingale.

We recall the **Doob decomposition** of supermartingale; every integrable supermartingale $(Z_n)_{n\in N}$ can be written in a unique way as the difference of an integrable martingale $(M_n)_{n\in N}$ and an increasing process $(B_n)_{n\in N}$, say $Z_n = M_n - B_n$.

Further uniform integrability of $(Z_n)_{n\in N}$ is equivalent to regularity of the martingale $(M_n)_{n\in N}$ together with the condition $B_\infty \in L'$ (see Neveu(1975) Proposition VIII-1-2).

The following our main result shows the structure of the largest optimal stopping time.

Theorem 3. Let $(X_n)_{n\in N}$ be a sequence of integrable random variables such that

$E(\sup_{n\in\bar{N}} X_n^+)<+\infty$. Let $(Z_n)_{n\in\bar{N}}$ be the Snell Envelope of $(X_n)_{n\in\bar{N}}$ and $Z_n=M_n-B_n$ the Doob decomposition of supermartingale $(Z_n)_{n\in N}$.

Then the stopping time T defind by

$$T(\omega)=\inf\{n : B_{n+1}(\omega)>0\}$$

is the largest optimal stopping time i.e., T is an optimal stopping time and $S\leq T$ for any optimal stopping time S.

Proof. We first show that supermartingale $(Z_n)_{n\in\bar{N}}$ is uniformly integrable. The inequality $E(Z_\infty|\mathscr{F}_n)\leq Z_n$ implies that $(Z_n^-)_{n\in\bar{N}}$ is uniformly integrable.

By our hypothesis we have $Z_n^+\leq E(\sup_{m\in\bar{N}} X_m^+|\mathscr{F}_n)$, which implies uniformly integrability of $(Z_n^+)_{n\in\bar{N}}$. Thus $(Z_n)_{n\in\bar{N}}$ is uniformly integrable. Next we will show that T is an optimal stopping time. We know that $B_{n+1}=0$ on the event $\{T>n\}$ by the definition of the stopping time T. Consequently $Z_{n+1}=M_{n+1}$ on $\{T>n\}$.

As a result,

$$\begin{aligned}
&E(Z_{n+1}|\mathscr{F}_n)1_{\{T>n\}}=E(Z_{n+1}1_{\{T>n\}}|\mathscr{F}_n)\\
&=E(M_{n+1}1_{\{T>n\}}|\mathscr{F}_n)=E(M_{n+1}|\mathscr{F}_n)1_{\{T>n\}}\\
&=M_n1_{\{T>n\}}=Z_n1_{\{T>n\}}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&E(Z_{T\wedge(n+1)}|\mathscr{F}_n)=E(Z_T1_{\{T\leq n\}}+Z_{n+1}1_{\{T>n\}}|\mathscr{F}_n)\\
&=Z_T1_{\{T\leq n\}}+E(Z_{n+1}|\mathscr{F}_n)1_{\{T>n\}}\\
&=Z_T1_{\{T\leq n\}}+Z_n1_{\{T>n\}}\\
&=Z_{T\wedge n}.
\end{aligned}$$

The above equation shows that $(Z_{T\wedge n})_{n\in N}$ is an integrable martingale. From the previous remark $(Z_{T\wedge n})_{n\in N}$ is uniformly integrable martingale and thus $(Z_{T\wedge n})_{n\in\bar{N}}$ is a martingale i.e., $E(Z_T|\mathscr{F}_n)=Z_{T\wedge n}$. Next we claim that $X_T=Z_T$. On the event $\{T=n\}$ we see that $B_n=0$ and $B_{n+1}>0$ by the definition of the stopping time T.

Thus we have

$$\begin{aligned}
&E(Z_{n+1}|\mathscr{F}_n)1_{\{T=n\}}=E(Z_{n+1}1_{\{T=n\}}|\mathscr{F}_n)\\
&=E((M_{n+1}-B_{n+1})1_{\{T=n\}}|\mathscr{F}_n)=E(M_{n+1}1_{\{T=n\}}|\mathscr{F}_n)-B_{n+1}1_{\{T=n\}}\\
&=(M_n-B_{n+1})1_{\{T=n\}}<M_n1_{\{T=n\}}=Z_n1_{\{T=n\}}.
\end{aligned}$$

From the equality $Z_n=\max(X_n,\ E(Z_{n+1}|\mathscr{F}_n))$ and the above inequality, we have $Z_n=X_n$ on $\{T=n\}$. Clearly we have $Z_\infty=X_\infty$ on $\{T=\infty\}$. Therefore $Z_T=X_T$ is obtained. Thus Lemma 2 shows that T is an optimal stopping time. It remains to show that T is the largest optimal stopping time. Let S be any optimal stopping time.

Then by Lemma 2 we have that $X_S = Z_S$ and $(Z_{S\wedge n})_{n\in N}$ is a martingale.

Thus for each $n \in N$

$$E(Z_0) = E(Z_{S\wedge n}) = E(M_{S\wedge n} - B_{S\wedge n})$$
$$= E(M_{S\wedge n}) - E(B_{S\wedge n}).$$

On the other hand, from the uniform integrability of (Z_n) we know that $(M_n)_{n\in N}$ is a regular martingale. Therefore $E(M_{S\wedge n}) = E(M_0)$.

Thus we conclude that $E(M_0) = E(Z_0) = E(M_{S\wedge n}) - E(B_{S\wedge n}) = E(M_0) - E(B_{S\wedge n})$ This shows that $E(B_{S\wedge n}) = 0$ and hence $B_{S\wedge n} = 0$ because $B_{S\wedge n} \geq 0$. By the definition of the stopping time T, we have $S\wedge n \leqq T$ for all n and hence we have $S \leq T$.

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