⟨Research Paper⟩

Journal of the Korean Statistical Society
Vol. 12, No. 1, 1983

Weak Convergence of Processes Occurring in Statistical Mechanics**

Jong Woo Jeon*

ABSTRACT

Let $X_j^{(n)}$, $j=1,2,\cdots n$, $n=1,2,\cdots$ be a triangular array of random variables which arise naturally in a study of ferromagnetism in statistical mechanics. This paper presents weak convergence of random function $W_n(t)$, an appropriately normalized partial sum process based on $S_n^{(n)}=X_1^{(n)}+\cdots+X_n^{(n)}$. The limiting process $W(t)$ is shown to be Gaussian when weak dependence exists. At the critical point where the change from weak to strong dependence takes place, $W(t)$ turns out to be non-Gaussian. Our results are direct extensions of work by Ellis and Newman (1978). An example is considered and the relation of these results to critical phenomena is briefly explained.

1. Introduction

Let P be a probability measure on the real line satisfying

$$\int \exp\left(\frac{x^2}{2}\right) p(dx) < \infty \tag{1.1}$$

Let $X_j^{(n)}$, $j=1,\cdots,n$, $n=1,2,\cdots$ be the triangular array of random variables with the joint distribution Q_n defined by

$$Q_n(d\underline{x}) = d_n^{-1} \exp \ (s_n^2/2n) \prod_{j=1}^{n} p(dx_j) \tag{1.2}$$

where $\underline{x}=(x_1,\cdots,x_n)$, $s_n=x_1+\cdots+x_n$ and d_n is a normalizing constant which is finite for each $n\geq 1$ in view of (1.1).

The model (1.1) is usually called the mean field model or the Curie-Weiss model in the statistical mechanics literature. The Curie-Weiss model has been considered important

*Department of Computer Science & Statistics, Seoul National University
**This work was supported by the Ministry of Education, through the Research Institute of Basic Sciences, Seoul National University 1982—1983.

due to the fact that it explains qualitatively and correctly the thermodynamic behavior of some physical quantities in the neighborhood of the critical point. See Stanley (1975) for reference.

In recent years, a probabilistic approach for the theory of the Curie-Weiss model has been developed. A typical result is to relate the validity or non-validity of the central limit theorem for $S_n^{(n)}$ to the non-criticality or criticality of phase. References along this line are Simon and Griffiths (1973) and Dunlop and Newman (1975). The latest result on the asymptotic distribution of $S_n^{(n)}$ was obtained by Ellis and Newman (1978).

In this paper, we consider $W_n(t)$, a normalized partial sum process based on $S_n^{(n)}$ and establish weak convergence of $W_n(t)$. Analogous to the previous results, the limiting process $W(t)$ is shown to be Gaussian at the non-critical phase. We note that this limiting Gaussian process has the property that increments are not independent. At the critical phase, $W(t)$ turns out to be non-Gaussian as expected. These results directly extend the works of Ellis and Newman. As for the proof, we utilize the particular form appearing in (1.2) to apply the conditioning technique. This method of proof is well explained in Jeon and Sethuraman (1983).

In Section 2, we present the results of Ellis and Newman together with some preliminaries. The main results are stated and proved in Section 3. Section 4 is devoted for an example. Finally, the relation of these results to critical phenomena is briefly discussed in Section 5.

2. Preliminaries

In this section, we state some of the results presented in Ellis and Newman (1978) as preliminaries. The detail can be consulted by their paper.

Let $M(t)=\int\exp(tx)\ P(dx)$ be the moment generating function (m.g.f.) of P. Let $L(t)=\log M(t)$. Let the function G be defined by

$$G(t)=\frac{t^2}{2}-L(t). \tag{2.1}$$

Lemma 2.1 The function G is real analytic and $G(t)\to\infty$ as $|t|\to\infty$. Thus, G has only a finite number of global minima. Also

$$\int\exp\{-n\ G(t)\}dt<\infty \text{ for any } n=1,2,\cdots \tag{2.2}$$

Let m be a point of a global minimum of G. We call

$k=k(m)$ the type and $c=c(m)$ the strength of m if as $t\to m$,

$$G(t)=G(m)+\frac{c(t-m)^{2k}}{(2k)!}+o(|t-m|^{2k}). \tag{2.3}$$

We call the measure P pure if G has a unique global minimum.

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of random variables. We write $Y_n\to N(\mu,\sigma^2)$ or $Y_n\to f(t)$ if the distribution of Y_n converges weakly to the normal distribution with the mean μ and the variance σ^2 or to the distribution whose density function is proportional to $f(t)$.

We are now in position to state the theorem of Ellis and Newman.

Theorem 2.2. Suppose that P is pure with the unique global minimum at m of type k and strength c. Then,

$$\frac{S_n^{(n)}-nm}{n^{1-1/2k}}\to\begin{cases}N(0,c^{-1}-1), & \text{if } k=1\\ \exp\{-ct^{2k}/(2k)!\}, & \text{if } k\geq 2\end{cases}$$

where $c^{-1}-1>0$ for $k=1$.

3. Main Results

For $0\leq t\leq 1$, let

$$W_n(t)=\frac{\sum_{i=1}^{[nt]}(X_i^{(n)}-m)+(X^{(n)}_{[nt]+1}-m)}{n^{1-1/2k}} \tag{3.1}$$

where $[nt]$ denotes the largest integer not exceeding nt. Then $W_n(1)=(S_n^{(n)}-nm)/n^{1-1/2k}$ and therefore has a limiting distribution as stated in Theorem 2.2. In this section, we prove that under those very conditions, the stochastic process $\{W_n(t), 0\leq t\leq 1\}$ converges weakly to a process $\{W(t), 0\leq t\leq 1\}$, where $W(\cdot)$ is non-Gaussian if $k\geq 2$ and is a Gaussian process which is not a Brownian motion if $k=1$.

3.1. Conditional Weak Convergence

To apply conditioning technique, we first introduce a fictitious random variable Z_n as follows: Rewrite Q_n in (1.2) as

$$\begin{aligned}Q_n(d\underline{x})&=d_n^{-1}\exp(s_n^2/2n)\prod_{j=1}^{n}p(dx_j)\\&=d_n^{-1}n^{(k-1)/2k}(2\pi)^{-1/2}\int\exp\{s_n(zn^{-1/2k}+m)-n(zn^{-1/2k}+m)^2/2\}\\&\quad\times dz\prod_{j=1}^{n}p(dx_j)\end{aligned}$$

$$=d_n^{-1}n^{(k-1)/2k}(2\pi)^{-1/2}\int\exp\{s_n(zn^{-1/2k}+m)-n\ L(zn^{-1/2k}+m)\}$$

$$\times\prod_{j=1}^{n}p(dx_j)\exp\{-nG(zn^{-1/2k}+m)\}dz$$

$$=d_n^{-1}n^{(k-1)/2k}(2\pi)^{-1/2}\exp\{-nG(m)\}$$

$$\int\prod_{j=1}^{n}\exp\{x_j(zn^{-1/2k}+m)-L(zn^{-1/2k}+m)\}$$

$$\times p(\ dx_j)\exp[-n\{G(zn^{-1/2k}+m)-G(m)\}]dz$$

$$=K_n^{-1}\int\prod_{j=1}^{n}R_{n,z}(dx_j)h_n(z)dz$$

$$=\int\prod_{j=1}^{n}R_{n,z}(dx_j)f_n(z)dz \tag{3.2}$$

where

$$R_{n,z}(dx)=\exp\{x(zn^{-1/2k}+m)-L(zn^{-1/2k}+m)\}p(dx) \tag{3.3}$$

$$h_n(z)=\exp[-n\{G(zn^{-1/2k}+m)-G(m)\}] \tag{3.4}$$

and

$$f_n(z)=h_n(z)/\int h_n(z)dz. \tag{3.5}$$

It becomes now obvious from (3.2) that $R_{n,z}$ is the common distribution of $X_j^{(n)}$, $j=1,\cdots,n$ conditional on $Z_n=z$ where f_n is the density function of Z_n.

Theorem 3.1. Let p be pure and of type k and strength c.

Then, under $R_{n,z}$, $\{W_n(t),\ 0\leq t\leq 1\}$ is tight.

Proof. Following the argument on page 60 of Billingsley (1968), we shall show that for each $\varepsilon>0$, there exist $\lambda>1$ and an integer n_0 such that if $n>n_0$, then

$$P_r\{\max_{i\leq n}|S_i^{(n)}-im|\geq\lambda_n^{1-1/2k}\}\leq\varepsilon/\lambda^2. \tag{3.6}$$

Now, since

$$n^{-1+1/2k}\max_{i<n}|iL'(zn^{-1/2k}+m)-im|$$

$$=n^{1/2k}|L'(zn^{-1/2k}+m)-L'(m)|$$

$$=n^{1/2k}|L''(m)zn^{-1/2k}+o(n^{-1/2k})|$$

$$=|L''(m)z+o(1)|,\ \text{as}\ n\to\infty,$$

there exists n_1 such that for $n>n_1$,

$$\max_{i\leq n}|iL'(zn^{-1/2k}+m)-im|<(3/2)L''(m)|z|n^{1-1/2k}. \tag{3.7}$$

Next, we note that, under $R_{n,z}$, the mean and the variance of $X_j^{(n)}$ are $L'(zn^{-1/2k}+m)$ and $L''(zn^{-1/2k}+m)$, respectively. It thus follows from the lemma on page 69 of Billingsley (1968) that

$$Pr\{\max|S_i^{(n)}-iL'(zn^{-1/2k}+m)|\geq(\lambda/2)n^{1-1/2k}\}$$

$$\leq 2\ Pr\left\{|S_n^{(n)}-nL'(zn^{-1/2k}+m)|\geq\left(\frac{\lambda n^{1-1/2k}}{2\sqrt{n\sigma_n^2}}-\sqrt{2}\right)\sqrt{n\sigma_n^2}\right\}$$

$$=2\,Pr\left\{\left|\frac{S_n^{(n)}-nL'(zn^{-1/2k}+m)}{\sqrt{n\sigma_n^2}}\right|\geq\frac{\lambda n^{(k-1)/2k}}{2\sigma_n}-\sqrt{2}\right\}$$

where

$$\sigma_n^2=L''(zn^{-1/2k}+m).$$

By the central limit theoem, there exists n_2 such that for $n>n_2$,

$$Pr\{\max|S_i^{(n)}-iL'(zn^{-1/2k}+m)|>(\lambda/2)n^{1-1/2k}\}$$

$$\leq 3\,Pr\left\{|N(0,1)|>\frac{\lambda n^{(k-1)/2k}}{2\,L''(m)^{1/2}}-\sqrt{2}\right\}$$

$$\leq 3\,Pr\left\{\ |N(0,1)|>\frac{\lambda n^{(k-1)/2k}}{2\cdot 2\cdot L''(m)^{1/2}}\right\}\ \text{for } \lambda>2\cdot 2\cdot(2L''(m))^{1/2}$$

$$\leq\frac{3\cdot 2^4\cdot 2^4\cdot\ L''(m)^2}{\lambda^4 n^{2(k-1)/k}}\ E|N(0,1)|^4 \tag{3.8}$$

Let $\varepsilon>0$ be given and choose λ and n_0 such that

$\lambda>\max\{L''(m)|z|,\ 48\,L''(m)/\varepsilon^{1/2}\}$ and $n_0=\max\{n_1,n_2\}$.

Then, for $n>n_0$,

$$Pr\{\max_{i\leq n}|S_i^{(n)}-im|\geq\lambda n^{1-1/2k}\}$$

$$\leq Pr\{\max_{i\leq n}|S_i^{(n)}-iL'(zn^{-1/2k}+m)|+\max_{i\leq n}|iL'(zn^{-1/2k}+m)-iL'(m)|\geq\lambda n^{1-1/2k}\}$$

$$\leq Pr\{\max_{i\leq n}|S_i^{(n)}-iL'(zn^{-1/2k}+m)|\geq\lambda n^{1-1/2k}/2\}$$

$$+Pr\{\max_{i\leq n}|iL'(m)-iL'(zn^{-1/2k}+m)|\geq\lambda n^{1-1/2k}/2\}$$

$$\leq\frac{3^2\cdot 2^4\cdot 2^4\cdot\ L''(m)^2}{\lambda^4 n^{2(k-1)/k}}+0\ \text{by (3.7) and (3.8)}$$

$$\leq\frac{3^2\cdot 2^4\cdot 2^4\ L''(m)^2}{\lambda^2}\cdot\frac{\varepsilon}{48^2\,L''(m)^2}=\varepsilon/\lambda^2.$$

This establishes(3. 6).

We now establish the finite dimensional convergence of $W_n(\cdot)$.

Theorem 3.2. Let p be pure and of type k and strength c.

Then, under $R_{n,z}$, for $0\leq s\leq t\leq 1$,

$$(W_n(s),\,W_n(t)-W_n(s))\to\begin{cases}\delta\{sz,\ (t-s)z\}, & \text{if } k\geq 2\\ N\left\{(sz(1-c),(t-s)\,z(1-c)),\begin{pmatrix}s(1-c) & 0\\ 0 & (t-s)(1-c)\end{pmatrix}\right\}, & \text{if } k=1.\end{cases}$$

where $\delta\{\cdot\}$ stands for degenerate distribution.

Proof. Since

$$\left|W_n(t)-\frac{S_{[nt]}^{(n)}-[nt]m}{n^{1-k/2}}\right|\leq\left|\frac{X_{[nt]+1}^{(n)}-m}{n^{1-1/2k}}\right|$$

and the left hand side in the above inequality converges to zero in probability for each $0 \leq t \leq 1$, it suffices to obtain the limiting distribution of

$$\left(\frac{S^{(n)}_{[ns]} - [ns]m}{n^{1-1/2k}}, \ \frac{(S^{(n)}_{[nt]} - S^{(n)}_{[ns]}) - ([nt]-[ns])m}{n^{1-1/2k}} \right). \tag{3.9}$$

It can be computed by using independence of $X_j^{(n)}$'s and was in fact shown in Jeon and Sethuraman (1983) that under $R_{n,z}$,

$$\frac{S^{(n)}_n - nm}{n^{1-1/2k}} \to \begin{cases} \delta\{z\} & , \quad \text{if } k \geq 2 \\ (N(1-c)z, \ (1-c)), & \text{if } k=1. \end{cases} \tag{3.10}$$

Accordingly, we have that

$$\frac{S^{(n)}_{[ns]} - [ns]m}{n^{1-1/2k}} \to \begin{cases} \delta\{sz\} & , \text{if } k \geq 2 \\ N(s(1-c)z, \ s(1-c)), & \text{if } k=1. \end{cases}$$

Since the components in (3.9) are independent, the theorem is proved.

We thus establish the following conditional weak convergence.

Theorem 3.3. Let P be pure and of type k and strength c. Then, under $R_{n,z}$, $W_n(\cdot)$ converges weakly to $W_z(\cdot)$ where $W_z(\cdot)$ is a Gaussian process with independent and stationary increments and with $EW_z(t) = zt(1-c)$ and Var $W_z(t) = (1-c)t$ if $k=1$. When $k \geq 2$, $W_z(\cdot)$ is a process degenerate at the function zt, $0 \leq t \leq 1$.

3.2. Weak Convergence

Recall that the joint distribution Q_n in (1.2) was reexpressed in (3.2) in terms of $R_{n,z}$ and f_n. Weak convergence of $W_n(\cdot)$ therefore follows from Theorem 2 of Sethuraman (1961), if $f_n(z)$ converges to a probability density function ($p.d.f.$) $f(z)$ for each z. That is to say that $W_n(\cdot)$ converges weakly for each fixed z (Theorem 3.3) and $f_n(z)$, the $p.d.f.$ of Z_n converges to a $p.d.f.$ $f(z)$ for each z. The limiting process $W(\cdot)$ will then be determined as the f-mixture of $W_z(\cdot)$ obtained in Theorem 3.3. Since the pointwise convergence of f_n to f which is proportional to $\exp(-cz^{2k}/(2k)!)$ was established in Jeon and Sethuramam (1983, Lemma 2.2), we obtain the following result of weak convergence fo $W_n(\cdot)$

Theorem. 3.4. Let P be pure and of type k and strength c. Then, $W_n(\cdot)$ converges weakly to a process $W(\cdot)$ where $W(\cdot)$ is a mixture of $W_z(\cdot)$ with $f(z)$. Its finite dimensional distribution is determined as follows:

For $0 \leq s \leq t \leq 1$,

$$(W(s), W(t) - W(s)) \sim N(\underline{0}, \ \Sigma = (\sigma_{ij})), \ \text{if } k=1$$

where

$$\sigma_{11}=s^2(1-c)^2c^{-1}+s(1-c),\quad \sigma_{22}=(t-s)^2(1-c)^2c^{-1}+(t-s)(1-c)$$
$$\sigma_{12}=\sigma_{21}=s(t-s)(1-c)^2c^{-1}$$

and

$$Pr\ \{W(s)<x,\ W(t)-W(s)<y\}=\int_{-\infty}^{u(x,y)} f(z)dz \text{ if } k\geq 2$$

where

$$u(x,y)=\min\{x/s,\ y/(t-s)\}$$

and

$$f(z)=\exp\{-cz^{2k}/(2k)!\}/\int\{\exp-cz^{2k}/(2k)!\}dz.$$

4. An Example

For $0<\beta\leq 1$, let P_β be the measure such that $P_\beta(\{\sqrt{\beta}\})=P_\beta(\{-\sqrt{\beta}\})=1/2$. Note that when $\beta=1$, P_β is the symmetric Bernoulli measure in which case the asymptotic distribution of $S_n^{(n)}$ was first considered by Simon and Griffiths (1973). Now, since

$$G_\beta(t)=t^2/2-\log\{\cos h(\sqrt{\beta t})\}>0\ \forall t\neq 0$$

and as $t\to\infty$,

$$G_\beta(t)=(1-\beta)t^2/2+3\beta^2t^4/(4!)+0(t),$$

P_β is pure with the unique global minimum at 0 and is of type 1 and strength $(1-\beta)$ if $\beta<1$ and is of type 2 and strength 3 if $\beta=1$. The limiting process $W(t)$ in this model can thus be obtained via Theorem 3.4. In particular, when $\beta=1$, we obtain the distribution of $W(1)$ as

$$\begin{aligned}&Pr\{W(1)\leq y\}\\&=Pr\{W(0)\leq x,\ \ W(1)-W(0)\leq y\}\\&=\int_{-\infty}^{y} f(z)dz\end{aligned}$$

where

$$f(z)=\exp(-z^4/12)/\int\exp(-z^4/12)dz.$$

The above coincides with the results of Simon and Griffiths (1973, Theorem 1) and Theorem 2.2 in Section 2.

5. Physical Interpretation

Suppose that we have a magnet which can be considered as a body consisting of an extremely large number of sites. The total magnetism of the body is the sum total of the magnetism present at the sites. The magnetism present at a site is also called a magnetic spin. To study the total magnetism in a body, several probabilistic model for magnetic spins have been proposed. A standard model states that

$$Pr((X_1^{(n)}\cdots, X_n^{(n)})\in d\underline{x})=Q_n(d\underline{x})$$

$$=d_n^{-1}e^{-\beta H(x_1,\dots,x_n)}\prod_{j=1}^{n} P(dx_j)$$

where $X_j^{(n)}$ represents the magnetic spin at the site j when there are n sites. The function $H(x_1, \cdots, x_n)$ is the Hamiltonian which represents the energy at the configuration $(x_1, \cdots, x_n)$ and $\beta > 0$ is the inverse temperature. In a ferromagnetic field, the form of H is given by $H(x_1, \cdots, x_n) = -1/2\sum J_{ij}x_ix_j$ where $J_{ij} > 0$. If, further, it is assumed that $J_{ij}=1/n$ for all i and j, that is to say, that each spin interacts equally with every other spin with strength $1/n$ and P is replaced by $P_\beta(x)=P(x/\sqrt{\beta})$, one obtains the Curie-Weiss model. The example in the previous section asserts that when $\beta<1$ (high temperature region), the central limit theorem holds whereas when $\beta=1$, it does not. In fact, under a different normalization it converges to a non-normal distribution. This reflects the physical fact that the critical temperature β_c at which phase transitions occur is equal to 1.

REFERENCES

(1) Billingsley, P. (1968). *Convergence of Probability Measures*, Wiley, New York.

(2) Dunlop, F. and Newman, C. M. (1975). Multicomponent field theories and classical rotators, *Commun. Math. Phys.*, Vol. 44, 223—235.

(3) Ellis, R. S. and Newman, C. M. (1978). Limit theorems for sums of dependent random variables occurring in statistical mechanics, *Z. F. Wahrscheinlichkeits-theorie Verw.*, Gebiete 44, 117—139.

(4) Jeon, J. W. and Sethuraman, J. (1983) Statistically motivated proofs of asymptotic distributions and phase transitions in the Curie-Weiss model, To appear *Proceedings of the 1st Saudi Symposium on Statistics and its Applications*.

(5) Sethuraman, J. (1961). Some limit theorems for joint distribution, *Sankhyā*, Ser. A, Vol. 23, 379—386.

(6) Simon, B. and Griffiths, R. B. (1973). The $(\phi^4)_2$ field theory as a classical Ising model, *Commun. Math. Phy.* Vol. 33, 145—164.

(7) Stanley, H. E. (1971). *Introduction to to Phase Transitions and Critical Phenomena*, Oxford, New York.