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A Study on the Bit Error Probability of PCM/FM System
(PCM/FM 시스템의 비트 誤差 確率에 關한 研究)

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要　約

本 論文에서는 PCM/FM시스템에 대한 비트(bit)誤差 確率式을 求하여 이를 수치적으로 計算하여 검토를 행했다.

變調前 필터를 거친 NRZ-L 二進 펄스列을 變調 信号로 하여 搬送波를 周波數 變調하였고, 리미터와 辨別器를 檢出器로 利用하였다.

RF 스펙트럼의 制限과 비트 誤差 確率의 감소面에서 볼 때, 周波數 偏移比 h를 3WT가 되도록 設計하는 것이 적당함을 알 수 있었다. 여기서 W는 變調前 필터의 帶域幅이고 T는 한 비트當 時間 間幅이다.

Abstract

The bit error probability of PCM/FM system was derived and numerically calculated. Binary NRZ-L baseband signal was premodulation filtered to frequency modulate a sinusoidal carrier, and a common circuit of limiter-discriminator was employed as a detector. Considering both RF spectrum limitation and bit error probability reduction, it was found that h ≒ 3WT would be reasonable, where h is the frequency deviation ratio, W is the bandwidth of a premodulation filter and T is the time interval of one bit.

I. Introduction

The bit error probability is commonly used as a measure of performance of digital communication systems. When the problem of baseband signal transmission is treated, the analysis becomes relatively easy [1, 2]. In the case of FM system, however, derivation of bit error probability is not so simple because of the inherent nonlinearity. Some studies on the digital FM systems can be found [3, 4], but the effect of a premodulation filter which is recommended to confine the radiated RF spectrum[5] has not been included in the studies.

In this paper, the bit error probability of PCM/FM telemetry system mounted on a vehicle to acquire various data during flight test was obtained, considering some system parameters frequency deviation ratio h and bandwidth of a premodulation filter W. The baseband signal assuming to be an NRZ-L (non-return-to-zero-level) binary pulse sequence was premodulation filtered and then

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frequency modulated a sinusoidal carrier. A common circuit of limiter-discriminator was used as a detection scheme.

Since the digital link was the main object to be studied, performance of limiter-discriminator circuit was assumed to be ideal. It was also assumed that the channel noise be additive white Gaussian, and there was no error due to the failure of bit synchronization.

II. System Modeling

The block diagram of PCM/FM system is shown in Fig. 1. It will be analysed and used to make some mathematical expressions. The output signal m(t) from binary data source is a serial pulses of NRZ−L data and can be described as

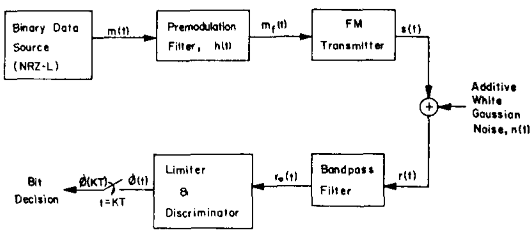

Fig. 1. Block diagram of PCM/FM system.

$$m(t) = \sum_{h=-\infty}^{\infty} b_n \; g[t-(n-1)T] \tag{1}$$

where b_n is the transmitted binary data that changes randomly between +1 and −1, and 1/T is bit rate, +1's and −1's are assumed to be equally likely. Using the unit step function u(t), g(t) is given by

$$g(t) = u(t) - u(t-T) \tag{2}$$

A premodulation filter is recommended by IRIG Standards[5] to confine the RF spectrum. If the impulse response of this filter is given by h(t), input signal $m_f(t)$ to the transmitter will be

$$m_f(t) = h(t) * m(t) = \sum_{n=-\infty}^{\infty} b_n \; x[t-(n-1)T] \tag{3}$$

where x(t)=h(t)*g(t).

Since the frequency of a FM transmitter output signal is proportional to the input signal, output signal s(t) of the transmitter will be

$$s(t) = \sqrt{2A} \; \cos\,[w_c t + \theta(t)] \tag{4}$$

where

$$\theta(t) = 2\pi f_d \int_{-\infty}^{t} m_f(\lambda)\, d\lambda. \tag{5}$$

Here, A is RF carrier power, w_c is the center radian frequency of carrier, and f_d is the maximum frequency deviation.

The purpose of the receiver bandpass filter is to remove out-of-band noise. Assuming that the bandpass filter has no effect on the signal itself, we can express the bandpass filter output $r_o(t)$ as the following equation,

$$r_o(t) = X(t)\cos w_c t - Y(t)\sin w_c t \tag{6}$$

where

$$X(t) = \sqrt{2A}\;\cos\,\theta(t) + n_x(t) \tag{7}$$

$$Y(t) = \sqrt{2A}\;\sin\,\theta(t) + n_y(t) \tag{8}$$

and $n_x(t)$, $n_y(t)$ are independent baseband Gaussian processes with zero means.

At the output of limiter-discriminator, it appears a signal equal to the instantaneous frequency of $r_o(t)$. Denoting it as $\dot{\phi}(t)$,

$$\dot{\phi}(t) = \frac{d}{dt}\left[\tan^{-1}\frac{Y(t)}{X(t)}\right] \tag{9}$$

$$= \frac{\dot{Y}(t)X(t) - Y(t)\dot{X}(t)}{X^2(t)+Y^2(t)}$$

where the dot represent differentiation with respect to time.

III. Bit Error Probability

Sampling the signal of (9) at t=kT, we can make bit decisions according to the sampled value $\dot{\phi}(kT)$. Thus the rule of bit decision can be made such as:

$$b_k = +1 \text{ was sent if } \dot{\phi}(kT) > 0$$
$$b_k = -1 \text{ was sent if } \dot{\phi}(kT) < 0 \tag{10}$$

Since the denominator of (9) is a sum of squares, the sign of (9) can be determined by the numerator only. If we denote the numberator of (9) as z(t),

$$z(t) = \dot{Y}(t)\,X(t) - Y(t)\,\dot{X}(t), \tag{11}$$

the bit error probability Pe can be expressed as

$$Pe = \tfrac{1}{2}\Big\{ \mathrm{Prob}\,[z(kT)<0|b_k=1] + \mathrm{Prob}\,[z(kT)>0\,|b_k=-1]\Big\}$$
$$= \tfrac{1}{2}\Big\{ \int_{-\infty}^{0} p(z/b_k=1)\,dz + \int_0^{\infty} p(z|b_k=-1)\,dz \Big\} \tag{12}$$

where p(z) is the probability density function of z(kT) that consists of four random variables; X(kT), Y(kT),$-\dot{X}$(kT) and $\dot{Y}$(kT).

Using the property that no pair of the four quantities $n_x(kT)$, $n_y(kT)$, $\dot{n}_x(kT)$, $\dot{n}_y(kT)$ is correlated[6], and the solution by Bennett and Davey[7], (12) can be simplified as follows.

$$Pe = \frac{1}{2\pi}\int_0^{\pi} \exp\left[-\frac{2\rho^2 a^2}{(1+a^2+k^2)+[(a^2+k^2-1)^2+4k^2]^{1/2}\cos\theta}\right] d\theta \tag{13}$$

where

$$\rho^2 = \frac{a_1^2 + a_2^2}{2\,\sigma_o^2} \tag{14}$$

$$a = \frac{a_1 a_4 + a_2 a_3}{a_1^2 + a_2^2}\,\frac{\sigma_0}{\sigma_1} \tag{15}$$

$$k = \frac{a_2 a_4 - a_1 a_3}{a_1^2 + a_2^2}\,\frac{\sigma_0}{\sigma_1} \tag{16}$$

and a_1, a_2, a_3, a_4 are mean of X(kT), Y(kT), $-\dot{X}$(kT), $\dot{Y}$(kT) respectively ;

$$a_1 = \sqrt{2A}\ \cos\,\theta(kT)$$
$$a_2 = \sqrt{2A}\ \sin\,\theta(kT)$$
$$a_3 = -\frac{d}{dt}\,[\sqrt{2A}\,\cos\,\theta(t)]\,|\,t=kT$$
$$a_4 = \frac{d}{dt}\,[\sqrt{2A}\,\sin\theta(t)]\,|\,t=kT. \tag{17}$$

In (15) and (16), σ_0^2 means the noise power of $n_x(t)$ or $n_y(t)$, and σ_1^2 means the power of $\dot{n}_x(t)$ or $\dot{n}_y(t)$. Therefore

$$\sigma_o^2 = \eta B \tag{18}$$

$$\sigma_1^2 = \frac{\pi^2\ \eta\ B^3}{3}. \tag{19}$$

where η is the one-sided power spectral density of the noise n(t), and B is the bandwidth of the receiver bandpass filter.

IV. Results and Discussions

1. Results

Substituting (17) into (14),

$$\rho^2 = \frac{A}{\sigma_o^2} \tag{20}$$

Since A is the carrier power and σ_0^2 is the noise power, ρ^2 means the carrier-to-noise ratio(CNR). And from (16) and (17), k=0.

In Fig. 2, the numerical values of bit error probability are plotted as a function of "a" for the three CNR cases (5dB, 10dB and 15dB). From (17), (18) and (19), "a" of (15) can be obtained as follows.

$$a = \dot{\theta}(kT)\frac{\sqrt{3}}{\pi B} = \sqrt{3}\ \frac{2fd}{B}\ m_f(kT) \qquad (21)$$

Finding $m_f(kT)$ from (3) and substituting it into (21),

$$a = \sqrt{3}\ \frac{2fd}{B}\left\{b_k x(T) + \sum_{\substack{n=-\infty \\ n \neq k}}^{\infty} b_n x[(k-n+1)T]\right\} \qquad (22)$$

where x(t) will be given by knowing the impulse response h(t) of the premodulation filter. Here, for the easy computations, we use a 2nd-order Butterworth lowpass filter as our optimum premodulation filter. Thus, denoting its frequency characterististic as H(jw),

$$H(jw) = \frac{(2\pi W)^2}{(jw)^2 + \sqrt{2}(2\pi W)\,jw + (2\pi W)^2} \qquad (23)$$

where W is the frequency at which the gain becomes −3dB, W is also used to determine the bandwidth B of the receiver bandpass filter. By Carson's rule, B can be determined as follows.

$$B = 2\,(W + f_d) \qquad (24)$$

In Table 1, the values of x(kT) were shown for the different WT's (WT = 0.3, 0.5, 0.7).

Now, knowing, from Table 1, that it would be sufficient to take into account the intersymbol interference from the five preceding pulses, and using (24), (22) can be shown as follows.

$$a = \sqrt{3}\ \frac{h}{h+2WT}\left\{b_k X(T) + \sum_{n=k-5}^{k-1} b_n x[(k-n+1)T]\right\} \qquad (25)$$

where $h=2f_dT$ is the frequency deviation ratio. Since x(kT) is a function of WT, "a" is expressed as a function of h, WT and five values of b_n. Five binary symbols will form 32 bit patterns, and so "a" will have 32 different values. For each of them, the bit error probability Pe/i can be calculated from (13). Pe/i is the bit error probability when the bit pattern i has been transmitted. Averaging these 32 Pe/i's will give the resultant bit error probability Pe ;

$$Pe = \tfrac{1}{2}\left\{\sum_{i=1}^{32} Pe/i\ \ Pi/_{b_{k=1}} + \sum_{i=1}^{32} Pe/i\ \ Pi/_{bk=-1}\right\} \qquad \textbf{(26)}$$

where Pi is the probability that the bit pattern i can occur, and was assumed to be 1/32.

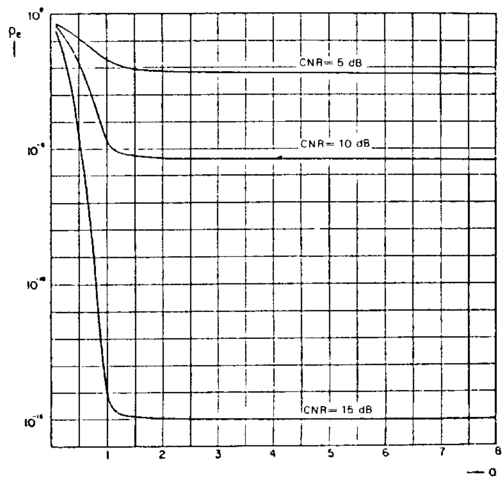

Fig. 2. Bit error probability versus "a."

The numerical values of the resultant bit error probability calculated from (13), (25) and (26) are shown in Fig. 3. And in the appendix, the numerical values of error probabilities with 32 bit patterns for the case of CNR=10dB (ρ^2= 10) and h=3WT are tabulated as Table 2.

Table 1. Values of x (kT) versus time-bandwidth products.

Time(KT)	WT = 0.3	WT = 0.5	WT = 0.7
T	0.6815517545	0.9793946147	1.0431699753
2T	0.3484093249	0.0350744799	- 0.0450296253
3T	- 0.0040887268	- 0.0161276925	0.0019396069
4T	- 0.0247396808	0.0017063046	- 0.0000833554
5T	- 0.0027911309	- 0.0000344792	0.0000035734
6T	0.0013736332	-0.0000155396	- 0.0000001528
7T	0.0003648813	0.0000024471	0.0000000065
8T	- 0.0000501742	- 0.0000001387	- 0.0000000003
9T	- 0.0000316144	- 0.0000000106	0.0000000000
10T	- 0.0000004406	0.0000000030	- 0.0000000000

2. Discussions

Fig. 2 and Fig. 3 show that the bit error probability decreases for the constant WT and h(or"a")when CNR increases. In data transmission systems, error probabilities of 10^{-5} are typical. Therefore, in order to maintain Pe below the order of 10^{-5}, it can be seen from Fig. 2 that CNR of 10dB is required at least. And from Fig. 3, for the constant CNR and WT, Pe decreases when the frequency deviation ratio h increases.

Note, however, that Pe hardly decreases if h exceeds some value. Larger values of h make Pe decreasing anyway, and yet also let B of (24) becoming larger. Since we have used the premodulation filter to confine the RF spectrum and B means the RF signal bandwidth, h being too large is against our intention. Here, we should make compromise for both the restriction of RF spectrum and the decrease of the bit error probability. Accordingly, from Fig. 2, it reminds us of the fact that the bit error probability Pe decreases rapidly for a less than 1, but hardly decreases for "a" greater than 1. Thus we might well select the value of h so that "a" will be approximately 1. If we let "a"=1 for example, the frequency deviation ratio h ≒ 2 (≒ 3WT) is obtained from (21) and (25) when WT=0.7 because $m_f(kT)$ ≒ 1 in that case. Namely, for WT=0.7 and h greater than 2, it can be said that the bit error probability hardly decreases while the required RF spectrum is getting larger.

From the results mentioned above, it can be concluded that h ≒ 3WT would be a good criterion for system design for both RF signal bandwidth and bit error probability reduction when W is large enough not to distort the baseband binary signal significantly.

Giving an example, a PCM/FM telemetry system that is used in a flight test of an unmanned vehicle has the bit rate 1/T=140KHz, maximum frequency deviation f_d=150KHz and premodulation filter bandwidth W=

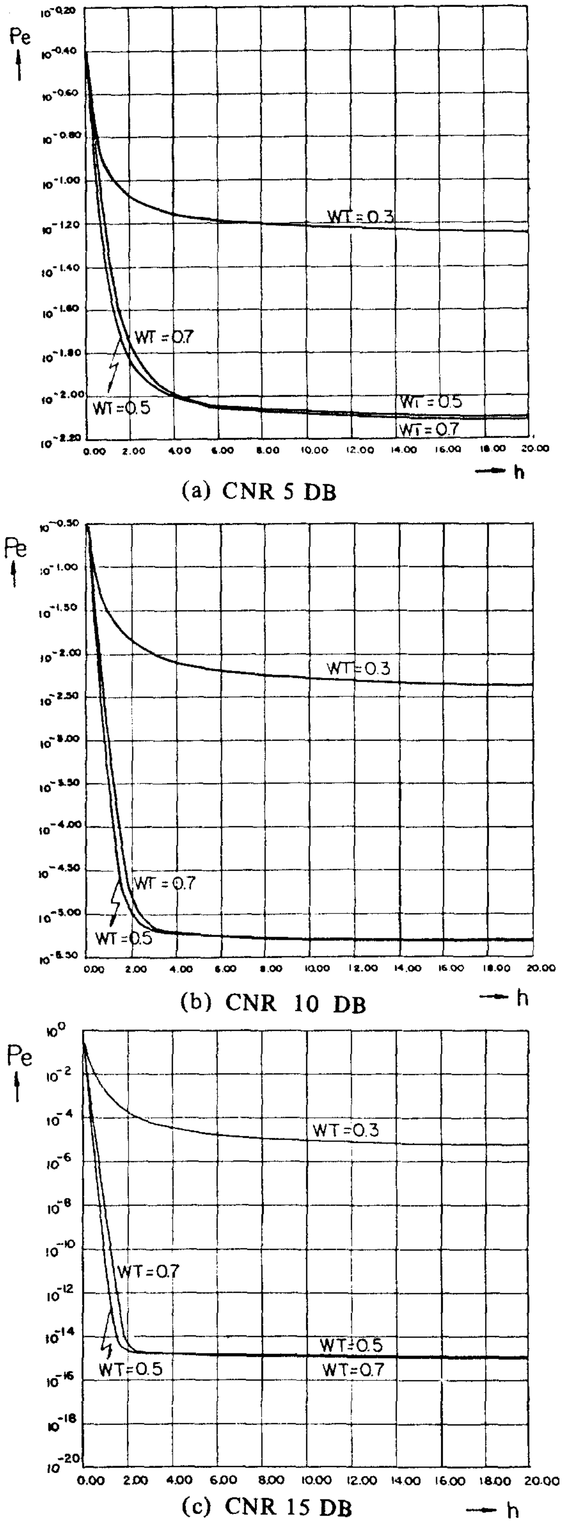

Fig. 3. Bit error probability versus h.

100KHz. We can see that this system satisfies the relation mentioned above very well since the deviation ratio $h=2f_dT \fallingdotseq 2$ and WT=0.7.

Appendix

Table 2. Bit error probabilities(Pe/i) according to 32 bit patterns when CNR = 10dB and h = 3WT.

Bit Patterns									
i	b_{k-5}	b_{k-3}	b_{k-3}	b_{k-2}	b_{-1}	b_k	WT=0.3	WT=0.5	WT=0.7
1	−1	−1	−1	−1	−1	+1	5.0×10^{-2}	2.4×10^{-5}	9.4×10^{-6}
2	−1	−1	−1	−1	+1	+1	5.4×10^{-2}	1.3×10^{-5}	1.6×10^{-5}
3	−1	−1	−1	+1	−1	+1	5.4×10^{-2}	3.4×10^{-5}	9.3×10^{-6}
4	−1	−1	−1	+1	+1	+1	1.1×10^{-5}	1.6×10^{-5}	1.6×10^{-5}
5	−1	−1	+1	−1	+1	+1	7.8×10^{-2}	2.3×10^{-5}	9.4×10^{-6}
6	−1	−1	+1	−1	+1	+1	1.5×10^{-5}	1.3×10^{-5}	1.6×10^{-5}
7	−1	−1	+1	+1	−1	+1	8.4×10^{-2}	3.3×10^{-5}	9.3×10^{-6}
8	−1	−1	+1	+1	+1	+1	1.6×10^{-5}	1.6×10^{-5}	1.6×10^{-5}
9	−1	+1	−1	−1	−1	+1	5.3×10^{-2}	2.4×10^{-5}	9.4×10^{-6}
10	−1	+1	−1	−1	−1	+1	5.3×10^{-2}	2.4×10^{-5}	9.4×10^{-6}
11	−1	+1	−1	+1	−1	+1	5.7×10^{-2}	3.4×10^{-5}	9.3×10^{-6}
12	−1	+1	−1	+1	+1	+1	1.2×10^{-5}	1.6×10^{-5}	1.6×10^{-5}
13	−1	+1	+1	−1	−1	+1	8.2×10^{-2}	2.3×10^{-5}	9.4×10^{-6}
14	−1	+1	+1	−1	+1	+1	1.5×10^{-5}	1.3×10^{-5}	1.6×10^{-5}
15	−1	+1	+1	+1	−1	+1	8.7×10^{-2}	3.3×10^{-5}	9.3×10^{-6}
16	−1	+1	+1	+1	+1	+1	1.6×10^{-5}	1.6×10^{-5}	1.6×10^{-5}
17	+1	−1	−1	−1	−1	+1	$4\ 9 \times 10^{-2}$	2.4×10^{-5}	9.4×10^{-6}
18	+1	−1	−1	−1	+1	+1	1.1×10^{-5}	1.3×10^{-5}	1.6×10^{-5}
19	+1	−1	−1	+1	−1	+1	5.3×10^{-2}	3.4×10^{-5}	9.3×10^{-6}
20	+1	−1	−1	+1	+1	+1	1.1×10^{-5}	1.6×10^{-5}	1.6×10^{-5}
21	+1	−1	+1	−1	1	+1	7.6×10^{-2}	2.3×10^{-5}	9.4×10^{-6}
22	+1	−1	+1	−1	+1	+1	1.4×10^{-5}	1.3 x	1.6×10^{-5}
23	+1	−1	+1	+1	−1	+1	1.4×10^{-5}	1.3×10^{-5}	1.6×10^{-5}
24	+1	−1	+1	+1	+1	+1	1.5×10^{-5}	1.6×10^{-5}	1.6×10^{-5}
25	+1	+1	−1	−1	−1	+1	5.2×10^{-2}	2.4×10^{-5}	9.4×10^{-6}
26	+1	+1	−1	−1	+1	+1	1.1×10^{-5}	1.3×10^{-5}	1.6×10^{-5}
27	+1	+1	−1	+1	−1	+1	5.6×10^{-2}	3.4×10^{-5}	9.3×10^{-6}
28	+1	+1	−1	+1	+1	+1	1.1×10^{-5}	1.6×10^{-5}	1.6×10^{-5}
29	+1	+1	+1	−1	−1	+1	8.0×10^{-2}	2.3×10^{-5}	9.4×10^{-6}
30	+1	+1	+1	−1	+1	+1	1.5×10^{-3}	1.3×10^{-5}	1.6×10^{-5}
31	+1	+1	+1	+1	−1	+1	8.6×10^{-2}	3.3×10^{-5}	9.3×10^{-6}
32	+1	+1	+1	+1	+1	+1	1.6×10^{-5}	1.6×10^{-5}	1.6×10^{-5}
average of Pe/i (Pe)							3.4×10^{-2}	2.1×10^{-5}	1.3×10^{-5}

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