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SOLUTION OF THE SUPER BESSEL WAVE EQUATION WITH INTEGRAL PARAMETER m

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Introduction.

Internal heat generation is one of the insidious conditions affecting the quality of an industrial product after it is cast, coated, molded, forged or laminated. Frequently, the product is pressed into service before the exothermic chemical reactions in the generic material has been completed. The heat liberated from this continuing chemical reaction or the residual deformation from the rheological activities in the material must be adequately removed or prevented, or the product may be discolored, warped, weakened or even "ignited"* spontaneously. Numerous instances of premature structural failures, product-recalls, and/or system-malfunctions have been recorded in recent history. The Coulee Dam was poured with pre-chilled concrete just to negate this freakish encore. It is well-known that concrete (a non-isotropic conducting medium), for instance, takes 28 days to develop its full strength. During this period of curing it is conceivable that the processes of internal heat generation, heat conduction and heat dissipation take place simultaneously inside the medium.

Assuming that the rate of internal heat generation is

$$q=q_0x^m(T-T_0),$$

where q_0, m(assumed to be an integer in this paper) are given constants, x is the spatial coordinate and $T-T_0$ is the temperature differential of the medium. The governing equation for the temperature distribution in the concrete during the transient phase of heat transfer is

$$C\rho\frac{\partial T}{\partial t}=K\frac{\partial^2 T}{\partial x^2}+q_0x^m(T-T_0) \tag{1}$$

where t is the time, T is the temperature, C is the specific heat, ρ is the density, K is the thermal conductivity, and T_0 is the ambient temperature; All material properties and the ambient temperature are assumed to be constant in this investigation.

The partial differential equation (1) is non-dimensionalized by introducing several dimensionless variables

$$\theta=\frac{T-T_0}{T_0},\quad \zeta=\frac{x}{L},\quad \tau=\frac{t}{C\rho L^2/K},\quad \beta^2=\frac{q_0L^{2+m}}{K}$$

where L is the thickness of the medium, thus

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial^2\theta}{\partial\zeta^2}+\beta^2\zeta^m\theta(\zeta,\tau)=0. \tag{2}$$

Assume that

$$\theta(\zeta,\tau)=T(\tau)X(\zeta)$$

*The term "ignition" here may mean non-existence of stable steady-state temperature distribution.

where $T(\tau)$ and $X(\zeta)$ are functions of τ and ζ, respectively.
Separation of variables in equation (2) yields

$$\frac{\dot{T}}{T}=\frac{X''}{X}+\beta^2\zeta^m=-\gamma^2 \tag{3}$$

where γ is a separation constant. We now have two ordinary differential equation from equation (3), namely,

$$\frac{\dot{T}}{T}=-\gamma^2 \tag{4}$$

and

$$\frac{X''}{X}+\beta^2\zeta^m=-\gamma^2. \tag{5}$$

The solution of equation (4) is of the form

$$T(\tau)=T_0e^{-\gamma^2\tau}$$

which is not an issue of this study.

We shall now solve the second order ordinary differential equation (5). Of course, we can use the method of Frobenius directly. But, we prefer another way to obtain its solution.

From equation (5), we have

$$X''+(\gamma^2+\beta^2\zeta^m)X=0.$$

Let $X=\zeta^{\frac{1}{2}}Y(\zeta)$, we abtain

$$\frac{d^2Y}{d\zeta^2}+\frac{1}{\zeta}\frac{dY}{d\zeta}+\left[\beta^2\zeta^m+\gamma^2-\frac{\left(\frac{1}{2}\right)^2}{\zeta^2}\right]Y(\zeta)=0. \tag{6}$$

The equation (6) is so called the Super Bessel Wave Equation on account of the presence of m, which may assume any interal value, including 2. If $m=2$, then the equation (6) becomes the Bessel wave equation,

$$\frac{d^2Y}{d\zeta^2}+\frac{1}{\zeta}\frac{dY}{d\zeta}+\left[\beta^2\zeta^2+\gamma^2-\frac{\left(\frac{1}{2}\right)^2}{\zeta^2}\right]Y(\zeta)=0. \tag{7}$$

The differential equation (7) has been solved for a more general order p in [1]. In this paper, we shall treat the simple case of m being an integer, but not necessarily 2.

Method of solution.

Equation (6) is tansposed

$$Y''(\zeta)+\frac{1}{\zeta}Y'(\zeta)+(\gamma^2-\frac{\frac{1}{4}}{\zeta^2})Y(\zeta)=-\beta^2\zeta^m Y(\zeta).$$

Now let $\xi=\gamma\zeta$, then

$$\frac{d^2Y}{d\xi^2}+\frac{1}{\xi}\frac{dY}{d\xi}+(1-\frac{\frac{1}{4}}{\xi^2})Y(\xi)=-\beta^2\frac{\xi^m}{\gamma^{m+2}}Y(\xi). \tag{8}$$

The first and second members of the complementary solution of equation (8) are

$$Y_1=\sqrt{\frac{2}{\pi}}\frac{\sin\xi}{\sqrt{\xi}} \quad\text{and}\quad Y_2=\sqrt{\frac{2}{\pi}}\frac{\cos\xi}{\sqrt{\xi}}.$$

The Wronskian of $Y_1(\xi)$ and $Y_2(\xi)$ is

$$W(\xi)=\begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix}=-\frac{2}{\pi\xi}$$

Then an application of the method of variation of parameters yields the following particular solution of equation (8).

$$Y_p(\xi)=-\sqrt{\frac{2}{\pi}}\frac{\cos\xi}{\sqrt{\xi}}\int_0^\xi\frac{\sqrt{\frac{2}{\pi}}\frac{\sin\xi'}{\sqrt{\xi'}}\left(-\frac{\beta^2}{\gamma^{m+2}}\xi'^m\right)Y(\xi')}{-\frac{2}{\pi\xi'}}d\xi'+\sqrt{\frac{2}{\pi}}\frac{\sin\xi}{\sqrt{\xi}}$$

$$\int_0^\xi\frac{\sqrt{\frac{2}{\pi}}\frac{\cos\xi'}{\sqrt{\xi'}}\left(-\frac{\beta^2}{\gamma^{m+2}}\xi'^m\right)}{-\frac{2}{\pi\xi'}}Y(\xi')d\xi'$$

$$=-\frac{\beta^2}{\gamma^{m+2}}\frac{1}{\sqrt{\xi}}\int_0^\xi\xi'^{m+\frac{1}{2}}(\cos\xi\sin\xi'-\sin\xi'\cos\xi')\,Y(\xi')d\xi'. \tag{9}$$

Multiplying both sides of equation (9) by $\sqrt{\xi}$, we obtain

$$\sqrt{\xi}\,Y_p(\xi)=-\frac{\beta^2}{\gamma^{m+2}}\int_0^\xi\xi'^m\sin(\xi-\xi')\sqrt{\xi'}\,Y(\xi')d\xi'.$$

The general solution of equation (8) is

$$Y(\xi)=C_1Y_1(\xi)+C_2Y_2(\xi)+Y_p(\xi),$$

so

$$Y_p(\xi)=Y(\xi)-\frac{A\sin\xi+B\cos\xi}{\sqrt{\xi}}.$$

Hence, we have

$$\sqrt{\xi}\,Y(\xi)-(A\sin\xi+B\cos\xi)=-\frac{\beta^2}{\gamma^{m+2}}\int_0^\xi\xi'^m\sin(\xi-\xi')\sqrt{\xi'}\,Y(\xi')d\xi'. \tag{10}$$

At this point, if we let m=0, we have

$$\sqrt{\xi}\,Y(\xi)-(A\sin\xi+B\cos\xi)=-\frac{\beta^2}{\gamma^2}\int_0^\xi\sin(\xi-\xi')\sqrt{\xi'}\,Y(\xi')d\xi'. \tag{11}$$

We now take Laplace tranforms of both sides of equation (11), resulting in

$$L\{\sqrt{\xi}\,Y(\xi)\}-\frac{A+BS}{1+S^2}=-\frac{\beta^2}{\gamma^2}\frac{1}{1+S^2}L\{\sqrt{\xi}\,Y(\xi)\}.$$

Transposing,

$$L\{\sqrt{\xi}\,Y(\xi)\}\left(1+\frac{\beta^2}{\gamma^2}\frac{1}{1+S^2}\right)=\frac{A+BS}{1+S^2}.$$

Therefore,

$$Y(\xi)=\frac{A\sin\sqrt{1+\left(\frac{\beta}{\gamma}\right)^2}\xi}{\sqrt{\xi}}+\frac{B\cos\sqrt{1+\left(\frac{\beta}{\gamma}\right)^2}\xi}{\sqrt{\xi}}.$$

For $m\neq0$, Laplace transformation of equation (11) gives

$$L\{\sqrt{\xi}\,Y(\xi)\}-\frac{A+BS}{1+S^2}=-\frac{\beta^2}{\gamma^{m+2}}\frac{1}{1+S^2}(-1)^m\frac{d^m}{dS^m}L\{\sqrt{\xi}\,Y(\xi)\}.$$

Let $y(S)=L\{\sqrt{\xi}\,Y(\xi)\}$, then

$$\frac{\beta^2}{\gamma^{m+2}}\frac{(-1)^m}{1+S^2}\frac{d^my(S)}{dS^m}+y(S)=\frac{A+BS}{1+S^2}$$

or

$$\frac{d^m y(S)}{dS^m}+\frac{\gamma^{m+2}}{\beta^2}(-1)^m(1+S^2)y(S)=\frac{\gamma^{m+2}}{\beta^2}(-1)^m(A+BS). \tag{12}$$

If $m=1$ equation (12) becomes

$$\frac{dy}{dS}-C_0(1+S^2)y=-C_0(A+BS), \quad \text{where} \quad C^0=\frac{\gamma^3}{\beta^2}.$$

Let

$$y=a_0+\frac{a_1}{S^1}+\frac{a_2}{S^2}+\frac{a_3}{S^3}+\frac{a_4}{S^4}\cdots\cdots \tag{13}$$

Substituting and equating the coefficients of like terms in the result, we obtain

$$a_0=0,\ a_1=B,\ a_2=A,\ a_3=-B,\ a_4=-\left(\frac{B}{C_0}+A\right),\ a_5=-\left(\frac{2A}{C_0}-B\right),$$

$$a_6=\frac{4B}{C_0}+A,\ a_7=\frac{4B}{C_0^2}+\frac{6A}{C_0}-B,\ a_8=\frac{10A}{C_0^2}+\frac{9B}{C_0}-A,\ a_9=-\left(\frac{28B}{C_0^2}+\frac{12A}{C_0}-B\right),\ \cdots\cdots$$

Thus,

$$y(S)=\frac{B}{S^1}+\frac{A}{S^2}-\frac{B}{S^3}-\frac{\frac{B}{C_0}+A}{S^4}+\frac{B-\frac{2A}{C_0}}{S^5}+\frac{\frac{4B}{C_0}+A}{S^6}+\cdots\cdots$$

and, by the definition of $y(S)$,

$$\sqrt{\xi}\,Y(\xi)=L^{-1}\left(\frac{B}{S^1}+\frac{A}{S^2}-\frac{B}{S^3}-\frac{\frac{B}{C_0}+A}{S^4}+\frac{B-\frac{2A}{C_0}}{S^5}+\frac{\frac{4B}{C_0}+A}{S^6}+\cdots\cdots\right).$$

Finally, we have

$$Y(\xi)=\frac{1}{\sqrt{\xi}}\left\{A\left(\xi-\frac{\xi^3}{3!}-\frac{2\xi^4}{C_0 4!}+\frac{\xi^5}{5!}+\cdots\cdots\right)+B\left(1-\frac{\xi^2}{2!}-\frac{\xi^3}{C_0 3!}+\frac{\xi^4}{4!}+\frac{4\xi^5}{C_0 5!}\cdots\cdots\right)\right\}$$

where $\xi=\gamma\zeta$.

If $m=2$, a similar procedure yields

$$y(S)=\frac{B}{S^1}+\frac{A}{S^2}-\frac{B}{S^3}-\frac{A}{S^4}+\frac{B-\frac{2B}{C_0}}{S^5}+\frac{A-\frac{6A}{C_0}}{S^6}-\frac{B-\frac{18B}{C_0}}{S^7}-\frac{A-\frac{26A}{C_0}}{S^8}$$
$$+\frac{B-\frac{44B}{C_1}-\frac{60B}{C_1^2}}{S^9}+\cdots\cdots,$$

where

$$C_1=\frac{\gamma^4}{\beta^2}.$$

Inversion gives

$$Y(\xi)=\frac{1}{\sqrt{\xi}}\left\{A\left(\xi-\frac{\xi^3}{3!}+\frac{1-\frac{6}{C_1}}{5!}\xi^5-\frac{1-\frac{26}{C^1}}{7!}\xi^7+\cdots\cdots\right)\right.$$
$$\left.+B\left(1-\frac{\xi^2}{2!}+\frac{1-\frac{2}{C_1}}{4!}\xi^4-\frac{1-\frac{18}{C_1}}{6!}\xi^6+\frac{1-\frac{44}{C_1}+\frac{60}{C_2}}{8!}\xi^8+\cdots\cdots\right)\right\}. \tag{14}$$

If $m=3$, we have

$$y(S)=\frac{B}{S^1}+\frac{A}{S^2}-\frac{B}{S^3}-\frac{A}{S^4}+\frac{B}{S^5}+\frac{A+\frac{6B}{C_2}}{S^6}-\frac{B+\frac{24}{C_2}A}{S^7}-\frac{A-\frac{54}{C_2}B}{S^8}+\frac{B+\frac{144}{C_2}A}{S^9}+\cdots\cdots$$

where

$$C_2=\frac{\gamma^5}{\beta^2}\text{ .}$$

Inversion gives for this case

$$Y(\xi)=\frac{1}{\sqrt{\xi}}\left\{A\left(\xi-\frac{\xi^3}{3!}+\frac{\xi^5}{5!}-\frac{24\xi^6}{C_2 6!}-\frac{\xi^7}{7!}+\frac{144\xi^8}{C_2 8!}+\cdots\cdots\right)\right.$$
$$\left.+B\left(1-\frac{\xi^2}{2!}+\frac{\xi^4}{4!}+\frac{6\xi^5}{C_2 5!}-\frac{\xi^6}{6!}+\frac{54\xi^7}{C_2 7!}+\frac{\xi^8}{8!}\cdots\cdots\right)\right\}.$$

One of the solutions of equation (8) for $m=2$ has been given by Moon and Spencer [1] and is called the solution of the Bessel wave equation

$$Y(\xi)=\frac{\sqrt{\frac{\xi}{2}}}{\Gamma\left(\frac{3}{2}\right)}\left\{1-\frac{\left(\frac{\xi}{2}\right)^2}{1!\left(\frac{3}{2}\right)}+\frac{\left(\frac{\xi}{2}\right)^4\left[1-4\left(\frac{3}{2}\right)\frac{\beta^2}{\gamma^2}\right]}{2!\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}-\frac{\left(\frac{\xi}{2}\right)^6\left[1-6\left(\frac{3}{2}+5\right)\frac{\beta^2}{\gamma^4}\right]}{3!\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)}+\cdots\cdots\right\}. \tag{15}$$

If we take $A=\frac{1}{\sqrt{2}\,\Gamma\left(\frac{3}{2}\right)}$ in equation (14), the first series in (14) and the series given by (15) become identical. In view of this agreement we may assume that our method for search for the solution of the Super Bessel Equation is valid for any integral m.

References

1. H. S. Carslaw and J. C. Jaegr, *Conduction of Heat in Solids,* Oxford Press, 1959, Second Edition,
2. T. M. Creese and R. M. Haralick, *Differential Equations for Engineers,* McGraw-Hill Book Company, 1978.
3. E. Jahnke and F. Emde, *Tables of Functions,* Dover Publications, 1945.
4. N. W. McLachlan, *Bessel Functions for Engineers,* Oxford Press, 1955.
5. P. Moon and D. E. Spencer, *Field Theory Handbook,* Springer-Verlag, 1961.
6. G. M. Murphy, *Ordinary Differential Equations and Their Solutions,* D. Van Nostrand Co., Inc., 1960.
7. E. D. Rainville, *Elementary Differential Equations,* The Macmilan Company, 1964.
8. S. L. Ross, *Differential Equations,* John Wiley & Sons, Inc., 1974.

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