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A NOTE ON S-CLOSED SPACES

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In this paper, we show a necessary and sufficient condition for QHC spaces to be S-closed.

T. Thomson introduced S-closed spaces in [2]. A topological space X is said to be *S-closed* if every semi-open cover of X admits a finite subfamily such that the closures of whose members cover the space, where a set A is *semi-open* if and only if there exists an open set U such that $U\subset A\subset Cl\ U$. A topological space X is *quasi-H-closed* (denote QHC) if every open cover has a finite subfamily whose closures cover the space. If a topological space X is Hausdorff and QHC, then X is H-closed.

It is obvious that every S-closed space is QHC but the converse is not true [2]. In [1], Cameron proved that an extremally disconnected QHC space is S-closed. But S-closed spaces are not necessarily extremally disconnected. Therefore we want to find a necessary and sufficient condition for QHC spaces to be S-closed.

A topological space X is said to be *semi-locally S-closed* if each point of X has a S-closed open neighborhood. Of course, a locally S-closed space is semi-locally S-closed.

THEOREM. *A topological space X is S-closed if and only if it is QHC and semi-locally S-closed.*

Proof. Since X is semi-locally S-closed, there is an open S-closed neighborhood U_x of x for each $x\in X$. Then $\{U_x : x\in X\}$ is an open cover of X. Since X is QHC, there is a finite subfamily $\{U_{x_i} : i=1, 2, \cdots n\}$ such that $X=\bigcup_{i=1}^{n} Cl\ U_{x_i}=Cl(\bigcup_{i=1}^{n} U_{x_i})$. X is S-closed by the following lemmas.

LEMMA 1. *If U is an open set in X which is S-closed in its relative topology, then $Cl\ U$ is also S-closed.*

Proof. Let $\{G_\alpha\}$ be a semi-open (in $Cl\ U$) cover of $Cl\ U$. We will show that $\{U\cap G_\alpha\}$ is also a semi-open cover of U. It is sufficient to show that for each α, we have

$$\mathrm{Int}_U(U\cap G_\alpha)\subset U\cap G_\alpha\subset Cl_U(\mathrm{Int}_U(U\cap G_\alpha)) \tag{1}$$

Since G_α is a semi-open set in $Cl\ U$, we have

$$\mathrm{Int}_{Cl\ U}G_\alpha\subset G_\alpha\subset Cl_{Cl\ U}(\mathrm{Int}_{Cl\ U}G_\alpha) \tag{2}$$

To prove the last implication of (1), we take any element $x\in U$ such that $x\notin Cl_U(\mathrm{Int}_U(U\cap G_\alpha))$. Then there exists a neighborhood N of x in U which is disjoint with $\mathrm{Int}_U(U\cap G_\alpha)$. Since U is an open set in X,

$$\mathrm{Int}_U(U\cap G_\alpha)=\mathrm{Int}_{Cl\ U}(U\cap G_\alpha) \tag{3}$$

and

$$\mathrm{Int}_{Cl\ U}(U\cap G_\alpha)=U\cap \mathrm{Int}_{Cl\ U}G_\alpha \tag{4}$$

Thus we obtain

$$\begin{aligned}\phi&=N\cap \mathrm{Int}_U(U\cap G_\alpha)=N\cap \mathrm{Int}_{Cl\ U}(U\cap G_\alpha)\\&=N\cap(U\cap \mathrm{Int}_{Cl\ U}G_\alpha)=N\cap \mathrm{Int}_{Cl\ U}G_\alpha.\end{aligned}$$

Since N is a neighborhood of x in U, it is also neighborhood of x in $Cl\ U$. Thus we have $x\notin Cl_{Cl\ U}(\mathrm{Int}_{Cl\ U}G_\alpha)$. By (2), we obtain $x\notin G_\alpha\cap U$. Thus $\{U\cap G_\alpha\}$ is a semi-open cover of U. Since U is S-closed, there is a finite subfamily $\{G_{\alpha_1}\cap U\ldots, G_{\alpha_n}\cap U\}$ of $\{G_\alpha\cap U\}$ such that $U=\bigcup_{i=1}^{n} Cl_U(G_{\alpha_i}\cap U)$. Therefore, we have a finite subfamily $\{G_{\alpha_i}: i=1,2,\cdots,n\}$ such that

$$\begin{aligned}Cl\ U=Cl_{Cl\ U}U&=Cl_{Cl\ U}(\bigcup_{i=1}^{n}Cl_U(G_{\alpha_i}\cap U))\\&=\bigcup_{i=1}^{n}Cl_{Cl\ U}(Cl_U(G_{\alpha_i}\cap U))\\&\subset\bigcup_{i=1}^{n}Cl_{Cl\ U}(G_{\alpha_i}\cap U)\\&\subset\bigcup_{i=1}^{n}Cl_{Cl\ U}G_{\alpha_i}.\end{aligned}$$

LEMMA 2. *If a topological space X is the union of finite number of open S-closed subspaces, then X is S-closed.*

Proof. Let U, V be open S-closed subspaces of X and $X=U\cup V$. Let $\{G_\alpha\}$ be a semi-open cover of X. Then $\{U\cap G_\alpha\}$ and $\{V\cap G_\alpha\}$ are also semi-open covers of U and V respectively. It is sufficient to show that

$$\mathrm{Int}_U(U\cap G_\alpha)\subset U\cap G_\alpha\subset Cl_U(\mathrm{Int}_U(U\cap G_\alpha))$$

If $x\in U$ and $x\notin Cl_U(\mathrm{Int}_U(U\cap G_\alpha))$, then there exists a neighborhood N of x in U such that $N\cap \mathrm{Int}_U(U\cap G_\alpha)=\phi$. Since $\mathrm{Int}_U(U\cap G_\alpha)=U\cap \mathrm{Int}\ G_\alpha$, we obtain

$$\phi=N\cap \mathrm{Int}_U(U\cap G_\alpha)=N\cap(U\cap \mathrm{Int}\ G_\alpha)=N\cap \mathrm{Int}\ G_\alpha.$$

Thus $x\notin Cl(\mathrm{Int}\ G_\alpha)$. Since $\mathrm{Int}\ G_\alpha\subset G_\alpha\subset Cl(\mathrm{Int}\ G_\alpha)$, x does not belong to G_α. Therefore we have $x\notin G_\alpha\cap U$. Since U, V are S-closed,

$$\begin{aligned}U=\bigcup_{i=1}^{m}Cl_U(U\cap G_{\alpha_i})&=\bigcup_{i=1}^{m}(Cl(U\cap G_{\alpha_i})\cap U)\\&\subset\bigcup_{i=1}^{m}Cl\ G_{\alpha_i},\end{aligned}$$

and

$$V=\bigcup_{j=1}^{n} Cl_V(U\cap G_{\alpha_j})=\bigcup_{j=1}^{n}(Cl(U\cap G_{\alpha_j})\cap V)$$
$$\subset\bigcup_{j=1}^{n} Cl\ G_{\alpha_j}.$$

Therefore X is S-closed.

References

1. Douglas E. Cameron, *Properties of S-closed spaces*, Proc. Amer. Math. Soc. **72** (1978), 581-586.
2. Travis Thompson, *S-closed spaces*, Proc. Amer. Math. Soc. **60**(1976), 335-338.

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