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ON A CLASS OF WEAKLY CONTINUOUS OPERATORS

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1. Introduction

Let X and Y be normed linear spaces. An operator T defined on X with the range in Y is continuous in the sense that if a sequence $\{x_n\}$ in X converges to x for the weak topology $\sigma(X, X')$ then $\{Tx_n\}$ converges to Tx for the norm topology in Y. We shall denote the class of such operators by $WC(X, Y)$.

For example, if T is a compact operator then $T\in WC(X, Y)$.

In this note we discuss relationships between $WC(X, Y)$ and the class of weakly compact operators $WK(X, Y)$, the class of compact operators $K(X, Y)$ and the class of bounded linear operators $B(X, Y)$. In the last section, we will consider some characters for an operator in $WC(X, Y)$.

2. Structures of $WC(X, Y)$

The class of compact operators $K(X, Y)$ is a linear subspace of $B(X, Y)$, but not closed unless Y is complete. There are some closed subspaces of $B(X, Y)$ see [6. p. 86], we add one more closed subspace.

PROPOSITION 2.1. *Let X be a normed linear space, Y a Banach space. Then $WC(X, Y)$ is a closed linear subspace of $B(X, Y)$ containing $K(X, Y)$ for the relative topology of the uniform operator topology τ_N in $B(X, Y)$.*

Proof. Obviously, $WC(X, Y)$ is a linear subspace of $B(X, Y)$ and $K(X, Y) \subseteq WC(X, Y)\subseteq B(X, Y)$. Let $\{T_m\}$ be a sequence in $WC(X, Y)$ such that $T_m\to T$ for the relative topology of τ_N in $B(X, Y)$. Then we have following inequalities:

$$\begin{aligned}\|Tx_n-Tx\| &\leq \|Tx_n-T_mx_n\|+\|T_mx_n-T_mx\|+\|T_mx-Tx\| \\ &\leq \|T-T_m\|\|x_n\|+\|T_mx_n-T_mx\|+\|T_m-T\|\|x\|.\end{aligned}$$

Moreover, it is known that a $\sigma(X, X')$-convergent sequence is norm bounded; (for the proof, since $\{x_n\}$ is weakly bounded there is a positive number $M>0$ such that

$$|f(x)|\leq M \text{ for any } x\in\{x_n\} (=S) \text{ and for every } f\in X'.$$

Let $J: X\to X''$ be the canonical embedding, then

$$\sup_{x\in S}|J(x)f| = \sup_{x\in S}|f(x)|<\infty \text{ for any } f\in X'.$$

Since J is an isometric isomorphism, the uniform bounded principle implies that $\sup_{x\in S}\|x\|=\sup_{x\in S}\|Jx\|<\infty$. Hence, $\{x_n\}=S$ is norm bounded.)

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Therefore, if $x_n \to x$ for $\sigma(X, X')$ then $Tx_n \to Tx$ for the norm in Y, thus $T \in WC(X, Y)$.

We denote $WC(X, X) = WC(X)$.

PROPOSITION 2.2. *Let X be a normed linear space. Then $WC(X)$ is a subalgebra of $B(X)$ and $WC(X)$ is a left ideal of $B(X)$.*

The proofs are immediate from the definition.

We notice that the identity operator is not contained in $WC(X)$, but, for special cases, it may happen that $I \in WC(X, Y)$ so that $WC(X, Y) = B(X, Y)$. For instance, if X is finite dimensional then the weak topology coincides with the norm topology in X, whence $WC(X, Y) = B(X, Y)$.

Another example, in the sequence space l^1, the weak convergence is identical to the norm convergence even if the weak topology and the norm topology are not coincide ([5], p. 157). Therefore if $X = l^1$ and Y a normed linear space, then $WC(X, Y) = B(X, Y)$.

The class $WC(X, Y)$ and the class $WK(X, Y)$ have no inclusion relation in general. It is known that every weakly compact operator is bounded, whence $WK(X, Y) \subseteq B(X, Y)$. And if at least one of X or Y is reflexive Banach space then every bounded operator is weakly compact. Thus if X or Y is reflexive, then $WK(X, Y) = B(X, Y)$.

For any normed linear spaces X and Y, obviously the following inclusion relation hold:

$$K(X, Y) \subseteq WK(X, Y) \subseteq B(X, Y)$$

and

$$K(X, Y) \subseteq WC(X, Y) \subseteq B(X, Y).$$

A relationship between an operator $T \in WC(X, Y)$ and a dual operator T' is given by the following proposition.

THEOREM 2.3. *Let X be a reflexive Banach space, and Y a normed linear space, then*

$$T \in WC(X, Y) \text{ implies that } T' \in WC(Y', X').$$

If both X and Y are reflexive, then

$$T \in WC(X, Y) \text{ if and only if } T' \in WC(Y', X').$$

Proof. It is known that if X and Y are normed linear spaces then $T \in K(X, Y)$ implies that $T' \in K(Y', X')$; and if Y is complete, then

$$T' \in K(Y', X') \text{ implies that } T \in K(X, Y).$$

Moreover, it can be shown that if $T \in K(X, Y)$ then $Tx_n \to Tx$ for the norm in Y whenever $x_n \to x$ for $\sigma(X, X')$. In case that X is reflexive then every $T \in WC(X, Y)$ is a compact operator. It follows that $K(X, Y) = WC(X, Y)$ whenever X is reflexive.

Therefore, if X is reflexive and Y a normed linear space, then

$$T \in WC(X, Y) = K(X, Y) \Rightarrow T' \in K(Y', X') \subseteq WC(Y', X').$$

$$\text{i.e. } T \in WC(X, Y) \Rightarrow T' \in WC(Y', X').$$

Suppose that both X and Y are reflexive. $K(Y', X') \subseteq WC(Y', X')$ as noted above, and since Y is reflexive if and only if Y' is reflexive, we have

$$K(Y', X') = WC(Y'X').$$

Thus

$$T' \in WC(Y', X') = K(Y', X') \Rightarrow T \in K(X, Y).$$

And since $K(X, Y) = WC(X, Y)$, we have

$$T' \in WC(Y', X') \Rightarrow T \in WC(X, Y).$$

COROLLARY 2.4. *Let H, W be Hilbert spaces, Then $T \in WC(H, W)$ if and only if $T' \in WC(W', H')$.*

This follows directly from Theorem 2.3.

In Propositions 2.1, 2.2 we have shown that $WC(X)$ is a closed left ideal in $B(X)$ whenever X is a Banach space. But this is not a right ideal in $B(X)$ in general. However, if X is reflexive then $WC(X)$ is a closed two sided ideal in $B(X)$; for, $K(X) = WC(X)$ and $K(X)$ is a closed two sided ideal of $B(X)$. Thus we have the following.

COROLLARY 2.5. *If X is a reflexive Banach space then $WC(X)$ is a closed two sided ideal in $B(X)$.*

For an adjoint operator, we have a following corollary.

COROLLARY 2.6. *Let H, W be Hilbert spaces then*

$$T \in WC(H, W) \text{ if and only if } T^* \in WC(W, H).$$

Proof. This follows from the facts that

$$T \in B(H, W) \text{ if and only if } T^* \in B(W, H),$$

and

$$T \in K(H, W) \text{ if and only if } T^* \in K(W, H).$$

Furthermore,

$$K(H, W) = WC(H, W), \quad K(W, H) = WC(W, H).$$

3. Characters of an operator in $WC(X, Y)$.

We list here a definition of the boundary which is related to the theory of Choquet boundary:

Let X be a compact Hausdorff space, a linear subspace A of $C(X)$ such that A separates the points of X and contains constants. A subset Y of X is called a "boundary" for A if for every $f \in A$ there exists $y \in Y$ such that $|f(y)| = \|f\|$.

The following theorem gives a generalized notion of the above mentioned boundary:

THEOREM 3.1. *Let X be a reflexive Banach space. For each $T\in WC(X, Y)$, there exists a unit vector $x\in X$ such that $\|T\|=\|Tx\|$.*

And if $T\in WC(X)$, there is a unit vector $f\in X'$ such that $\|T'\|=\|T'f\|$.

Proof. By definition of the norm of T in $B(X, Y)$, there is a sequence $\{x_n\}$ in $S=\{x\in X : \|x\|\leq 1\}$ such that $\|T\|=\lim_{n\to\infty}\|Tx_n\|$.

It is known that if X is reflexive, each bounded sequence in X contains a weakly convergent subsequence; in particular if $\{x_n\}$ is a sequence for which $\|x_n\|\leq 1$ then it contains a subsequence $\{x_m\}$ converging weakly to a limit x for which $\|x\|\leq 1$. Thus for a $T\in WC(X, Y)$, $Tx_m\to Tx$ for the norm whenever $x_m\to x$ for $\sigma(X, X')$, whence $\|Tx_m\|\to\|Tx\|$.

It follows that

$$\|T\|=\lim_{m\to\infty}\|Tx_m\|=\|Tx\|,\quad x\in S.$$

A simple calculation shows that $\|x\|=1$.

The second proposition follows from the fact that X is reflexive if and only if X' does, and since $T'\in WC(X')$ by Theorem 2.3, we apply the same arguments as the first part, there is a unit vector $f\in X'$such that $\|T'\|=\|T'f\|$. We have proved the theorem.

COROLLARY 3.2. *Let H and W be Hilbert spaces. For each T in $K(H, W)$ there is a unit vector x in H such that $\|T\|=\|Tx\|$, and a unit vector $f\in W'$ such that $\|T'\|=\|T'f\|$.*

Let X be reflexive, for any $T\in WC(X)$ and a positive integer n, $T^n\in WC(X)$ since $WC(X)$ is a subalgebra of $B(X)$. By Theorem 3.1 there is a unit, vector $x(n)$ such that $\|T^n\|=\|T^n x(n)\|$ thus the sequence $\{\|T^n x(n)\|\}^{1/n}$ converges as $n\to\infty$ and its limit is the spectral radius of T. If T is a proper contraction then the sequence $\{T^n x(n)\}$ converges to the zero vector.

LEMMA 3.3. *Let X be a reflexive Banach space, let $\mathcal{S}(T^n)=\{x\in\partial S : \|T^n\|=\|T^n x\|\}$ $(n\in N)$ for $T\in WC(X)$, where $\partial S=\{x\in X : \|x\|=1\}$.*

Then if there is a $p\in N$ for which $T^p=0$ then $\mathcal{S}(T^p)=\partial S$, while if there is no p for which $T^p=0$ then $\mathcal{S}(T^n)$ is a norm closed subset of ∂S for each $n\in N$.

Proof. The first part is obvious. Let $\{x_i\}$ be a sequence in $\mathcal{S}(T^n)$ such that $x_i\to x$ for the norm. Since $T^n\in WC(X)$, $\|T^n\|=\|T^n x_i\|\to\|T^n x\|$. Thus $\|T^n\|=\|T^n x\|$, $\|x\|=1$. Therefore $\mathcal{S}(T^n)$ is a closed subset of ∂S for each $n\in N$.

Obviously $\bigvee_{A\in WC(X)}\mathcal{S}(A)\ (=[\bigcup_{A\in WC(X)}\mathcal{S}(A)])=X$. We denote $U(A)=\{x\in\partial S : \|A\|\neq\|Ax\|\}$, then $U(0)=\phi$ and $\mathcal{S}(0)=\partial S$.

Whether or not does there exist A, B in $WC(X)$ for which $\mathcal{S}(A)\cap\mathcal{S}(B)=\phi$, or equivalently $U(A)\cup U(B)=\partial S$ so that $\bigvee_{A\in WC(X)}U(A)=X$. Instead of to answer on this question we shall show a restricted problem. To do this we begin with the following definition.

Let X be a complex Banach space. A closed subspace Y invariant under $T\in B(X)$

is said to be a spectral maximal space of T if it contains every closed subspace Z of X invariant under T with $\sigma(T|Z)\subset\sigma(T|Y)$. An operator T in $B(X)$ is decomposable if every finite open cover $\{G_i\}$ of the spectrum $\sigma(T)$, there exist a system of spectral maximal spaces $\{Y_i\}$ of T such that

(i) $\sigma(T|Y_i)\subset G_i$ $(i=1,2,\dots,n)$, (ii) $X=\sum_{i=1}^{n} Y_i$.

For a decomposable operator T, let σ be a separate part of $\sigma(T)$. then $E(\sigma,T)=\frac{1}{2\pi i}\int_\Gamma R_\lambda(T)d\lambda$ defines a projection operator, where Γ is any closed Jordan curve surounding σ and separating σ and $\sigma'=\sigma(T)\backslash\sigma$.

It is well known that

(1) $E(\sigma,T)X$ is a spectral maximal space of T, $(T|E(\sigma,T)X)=\sigma$, and (2) the restriction operator $T|E(\sigma,T)X$ is decomposable.

THEOREM 3.4. *Let X be a reflexive Banach space. Then for any nonzero operator T in $WC(X)$ there exist invariant subspaces X_1, X_2 under T such that*

(1) $X=X_1\oplus X_2$ (2) $T|X_i\in WC(X_i)\,(i=1,2)$ *and* (3) $\mathcal{S}(T|X_1)\cap\mathcal{S}(T|X_2)=\phi$.

Proof. Previously we noted that $WC(X)=K(X)$. It is known that every compact operator is decomposable ([4], p. 33), and the spectrum of a compact operator consists of at most countable number of eigenvalues of T, only limit point of it is 0. Therefore we may choose a disjoint open cover $\{G_1, G_2\}$ of $\sigma(T)$, that is,

$$\sigma(T)\subset G_1\cup G_2,\ G_1\cap G_2=\phi\ \ (G_i\neq\phi\ \ i=1,2)$$

putting $\sigma(T)\cap G_i=\sigma_i$, we have $\sigma_1\cup\sigma_2=\sigma(T)$ and $\sigma_1\cap\sigma_2=\phi$, obviously σ_i's are separate parts of $\sigma(T)$. Therefore $X_i=E(\sigma_i,T)X$ $(i=1,2)$ are spectral maximal spaces of T, whence X_i's are invariant subspaces of T. Furthermore it is easily seen that $X=X_1\oplus X_2$.

Since X is reflexive if and only if X_1, X_2 are reflexive, the restriction operators $T|X_i=T_i$ $(i=1,2)$ are elements of $WC(X_i)$ and both $\mathcal{S}(T_1)$ and $\mathcal{S}(T_2)$ are nonempty closed subsets of ∂S by Lemma 3.3.

Furthermore since $X=X_1\oplus X_2$, $\mathcal{S}(T_i)\subset X_i\cap\partial S$ $(i=1,2)$ and $\partial S\cap(X_1\cap X_2)=\phi$, we have $\mathcal{S}(T|X_1)\cap\mathcal{S}(T|X_2)=\phi$. This completes the proof.

From the last assertion of Theorem 3.4, we have the following corollary.

COROLLARY 3.5. *Let X be a reflexive Banach space. Then for any nonzero operator T in $WC(X)$ there exist invariant subspaces X_1 and X_2 under T such that $U(T|X_1)\vee U(T|X_2)=X$.*

Proof. Since $\mathcal{S}(T|X_1)\cap\mathcal{S}(T|X_2)=\phi$ by Theorem 3.4, $U(T|X_1)\cup U(T|X_2)=\partial S$. And since the closed linear span $[\partial S]$ of ∂S is the whole space X, we have the desired equality.

Let $\mathfrak{M}\subset B(X)$ be any set of operators and let $\mathfrak{X}\subset X$ be any set of vectors. We call $\mathfrak{X}$ separating for $\mathfrak{M}$ if $T\in\mathfrak{M}$ and $Tx=0$ for all $x\in\mathfrak{X}$ then $T=0$.

PROPOSITION 3.6. *Let X be reflexive. Then for each T in $WC(X)$ there is a set of separating vectors for $\{T\}$. If $\mathcal{S}(T)$ is cyclic for $WC(X)$ then $\mathcal{S}(T)$ separating for* *the commutant of* $WC(X)$.

Proof. The first assertion follows from Theorem 3.1. For the second, if S is an element of the commutant of $WC(X)$ and $Sx=0$ for all $x\in\mathcal{D}(T)$ then $S(Tx)=T(Sx)=0$. Since $\mathcal{D}(T)$ is cyclic for $WC(X)$, that is $[WC(X)\mathcal{D}(T)]=X$, we have $S=0$.

Let X, Y be normed linear spaces and let $N(T)$ be the null space of a bounded linear operator from X into Y. An operator $\hat{T}: X/N(T)\to Y$ defined by $\hat{T}[X]=Tx$ is norm continuous and $\|\hat{T}\|=\|T\|$, where $X/N(T)$ is the normed linear space with the quotient norm. For this operator, we have the following proposition.

PROPOSITION 3.7. *If* $\hat{T}\in WC(X/N(T), Y)$ *then* $T\in WC(X, Y)$, *and if* X *is reflexive then there is a unit vector* $[x]$ *in* $X/N(T)$ *such that* $\|\hat{T}\|=\|\hat{T}[x]\|$.

Proof. Since $(X/N(T))'\cong N(T)$, that is there is a natural isometric isomorphism of $(X/N(T))'$ onto $N(T)=\{f\in X' : f(x)=0,\ x\in N(T)\}$. Therefore, for each $F\in(X/N(T))'$ there exists a unique $f\in N(T)$ such that $F=f$ (identify). Thus

$$F([x_n])\to F(x) \text{ whenever } x_n\to x \text{ for } \sigma(X, X').$$

Hence $\hat{T}[x_n]$ converges to $\hat{T}[x]$ for the norm on Y, and since

$$\|\hat{T}[x_n]-\hat{T}[x]\|=\|Tx_n-Tx\|\to 0 \text{ as } n\to\infty,$$

we have $T\in WC(X, Y)$. Therefore, by Theorem 3.1, there is a unit vector $x\in X$ such that $\|T\|=\|Tx\|$. With this equality and with the fact that $\|\hat{T}\|=\|T\|$, $\hat{T}[x]=Tx$, we have the conclusions.

We notice that the reflexibility of $X/N(T)$ was dropped in the proposition 3.7, but the same conclution is obtained as in Theorem 3.1. However it is true that $X/N(T)$ is reflexive whenever X is reflexive.

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