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APPROXIMATION THEOREMS IN THE THEORY OF PSEUDODIFFERENTIAL OPERATORS

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In this paper we shall continue the study of the approximation theorems in the double pseudodifferential operators as in the single pseudodifferential operators.

1. Preliminaries

The class $S_{\rho,\delta}^m$ denote the Hörmander's class and the single pseudodifferential operators we consider are of the form

$$Au(x)=\frac{1}{(2\pi)^n}\iint a(x,\xi,y)u(y)e^{i(x-y)\cdot\xi}dyd\xi.$$

The amplitude $a(x,\xi,y)$ is assumed to belog to one of the following classes.

DEFINITION Let $0\leq\rho,\delta_1,\delta_2$. We say $a(x,\xi,y)\in S_{\rho,\delta_1,\delta_2}{}^m(\Omega\times\Omega\times R^n)$ if, on compact subsets of $\Omega\times\Omega$, we have

$$|D_y{}^\gamma D_x{}^\beta D_\xi{}^\alpha a(x,\xi,y)|\leq C(1+|\xi|^{m-\rho|\alpha|+\delta_1|\beta|+\delta_2|\gamma|}.$$

2. Multiple symbols

We now study operators of the form $b(D,x_1,D)$ and their associated multiple symbols.

DEFINITION 2.1. We say $b(\xi_2,x_1,\xi_1)\in S^{m_1,m_2}_{\rho,\delta_1,\delta_2}$ if for K compact we have

$$D_{x_1}{}^\beta D_{\xi_2}{}^\gamma D_{\xi_1}{}^\alpha b(\xi_2,x_1,\xi_1)|\leq C_{K,\alpha,\beta,\gamma}(1+|\xi_2|)^{m_2-\rho|\gamma|+\delta_2|\beta|}(1+|\xi_1|)^{m_1-\rho|\alpha|+\delta_1|\beta|}$$

for $x_1\in K$ and $\xi_i\in R^n$.

For convenience we shall assume that $b(\xi_2,x_1,\xi_1)=0$ for large x_1. Let

$$\hat{b}(\xi_2,n,\xi_1)=\int e^{-ix.n}b(\xi_2,x_1,\xi_1)dx_1.$$

DEFINITION 2.2. The double pseudodifferential operator $B=b(D,x_1,D)$ is defined by the formula

$$Bu(x)=\frac{1}{(2\pi)^{2n}}\iiint b(\xi_2,x_1,\xi_1)e^{ix\cdot\xi_2+iy\cdot(\xi_1-\xi_2)}\hat{u}(\xi_1)d\xi_1dy_1d\xi_2 \tag{2.1}$$

for $u\in\mathcal{S}$. Equivalent to (2.1) is the formula

$$(Bu)^\wedge(\xi_2)=\frac{1}{(2\pi)^n}\iint b(\xi_2,x_1,\xi_1)e^{iy\cdot(\xi_1-\xi_2)}\hat{u}(\xi_1)d\xi_1dx_1.$$
$$=\frac{1}{(2\pi)^n}\int\hat{b}(\xi_2,\xi_2-\eta,\eta)\hat{u}(\eta)d\eta.$$

Consequently, we can write

(2.2) $$b(D,x_1,D)u(x)=\frac{1}{(2\pi)^n}\int a(x,\xi_1)e^{ix\cdot\xi_1}\hat{u}(\xi_1)d\xi_1$$

where

(2.3) $$a(x,\xi_1)=\frac{1}{(2\pi)^n}\iint b(\xi_2,x_1,\xi_1)e^{i(x_1-x)\cdot(\xi_1-\xi_2)}dx_1d\xi_2$$
$$=\frac{1}{(2\pi)^n}\int\hat{b}(\xi_1+\eta,\eta,\xi_1)e^{ix\cdot\eta}d\eta$$

LEMMA 2.1. *Let* $p(\xi)\in S^m_{1,0}$, $-\infty<m<\infty$, *and* $b(\xi_2,x_1,\xi_1)=p(\xi_1)/p(\xi_2)$. *Then* $b(D,x_1,D):\mathcal{S}\to\mathcal{S}$ *is an identity.*

Proof. By the Fourier inversion formula,
$$\frac{1}{(2\pi)^n}\iint e^{ix\cdot(\xi_1-\xi_2)}p(\xi_1)\hat{u}(\xi_1)d\xi_1dx_1=p(\xi_2)\hat{u}(\xi_2).$$
Therefore we get
$$b(D,x_1,D)u(x)=\frac{1}{(2\pi)^n}\int e^{ix\cdot\xi_2}\hat{u}(\xi_2)d\xi_2=u(x).$$

3. Approximation Theorems for the double pseudodifferential operators

LEMMA 3.1. *Let* $p_j\in S^{m_j}_{\rho,\delta}(\Omega)$, $m_j\downarrow-\infty$, $j\geq0$. *Let* $p\in C^\infty(\Omega\times R^n)$ *and assume there are* $C_{\alpha\beta}$, $\mu=\mu(\alpha,\beta)$ *such that*
$$|D_x^\beta D_\xi^\alpha p(x,\xi)|\leq C_{\alpha\beta}(1+|\xi|)^\mu.$$
If there exist $\mu_k\to\infty$ *such that*
$$|p(x,\xi)-\sum_{j=0}^k p_j(x,\xi)|\leq C_k(1+|\xi|)^{-\mu_k},$$
then $p\in S^{m_0}_{\rho,\delta}(\Omega)$ *and* $p\sim\Sigma\, p_j$.

The proof may be found in Taylor [3]. Using this lemma, we analyse the amplitude $a(x,\xi)$ in (2.3).

THEOREM 3.2. *If* $b(\xi_2,x_1,\xi_1)\in S^{m_1,m_2}_{\rho,\delta_1,\delta_2}$ *has compact support in* x_1*-components, then* $a\in S^m_{\rho,\delta}$ *with* $m=m_1+m_2$, $\delta=\delta_1+\delta_2$, *provided* $0\leq\delta<\rho\leq1$. *Furthermore, we have the asymptotic expansion*

(3.1) $$a(x,\xi)\sim\sum_{|\alpha|\geq0}\frac{1}{\alpha!}D_x{}^\alpha\partial_{\xi_2}{}^\alpha b(\xi_2,x,\xi_1)|_{\xi_2=\xi_1}.$$

Proof. By Taylor's formula we have
$$\hat{b}(\xi+\eta,\eta,\xi_0)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\eta^\alpha\partial_{\xi_2}{}^\alpha\hat{b}(\xi_2,\eta,\xi)|_{\xi_2=\xi}+R_N(\eta,\xi_1)$$
where
$$R_N(\eta,\xi)=\frac{1}{(N-1)!}\int_0^1(1-t)^{N-1}\partial_t{}^N\hat{b}(\xi+t\eta,\eta,\xi)dt,$$
hence
$$|R_N(\eta,\xi)|\leq C_N\sup_{|r|=N,0\leq t\leq1}|D_{\xi_2}{}^r\hat{b}(\xi-t\eta,\eta,\xi)||\eta|^N.$$
Taking inverse Fourier transforms *w.r.t.* η yields

(3. 2) $$a(x,\xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}D_x{}^{\alpha}\partial^{\alpha}{}_{\xi_2}b(\xi_2,x,\xi)\,|_{\xi_2=\xi}+\tilde{R}_N(x,\xi)$$

where $\tilde{R}_N(x,\xi)=\frac{1}{(2\pi)^n}\int e^{ix\cdot\eta}R_N(\eta,\xi)\,d\eta$. The general term in the sum in (3. 2) clearly belongs to $S_{\rho,\delta}{}^{m-(\rho-\delta)|\alpha|}$. To complete the proof, we apply Lemma 3. 1. So it is only necessary to verify the following estimates:

(3. 3) $$|\tilde{R}_N(x,\xi)|\le C_N(1+|\xi|)^{m-(\rho-\delta)N},$$

(3. 4) $$|D_x{}^{\beta}D_\xi{}^{\alpha}a(x,\xi)|\le C_{\alpha\beta}(1+|\xi|)^{\mu(\alpha,\beta)}.$$

To prove (3. 3), note that, if $b\in S_{\rho,\delta_1,\delta_2}{}^{m_1,m_2}$, then

(3. 5) $$|D_{\xi_2}{}^{\gamma}\hat{b}(\xi_2,\eta,\xi_1)|\le C_{\gamma,\nu}(1+|\xi_2|)^{m_2-\rho|\gamma|+\delta_2\nu}(1+|\xi_1|)^{m_1+\delta_1\nu}(1+|\eta|)^{-\nu}.$$

Therefore

$$\begin{aligned}|R_N(\eta,\xi)|&\le C_N\sup_{|r|=N,0\le t\le 1}|D_{\xi_2}{}^{\gamma}\hat{b}(\xi+t\eta,\eta,\xi)|\,|\eta|^N\\&\le C_{N,\nu}\sup_{0\le t\le 1}(1+|\xi+t\eta|)^{m_2-\rho N+\delta_2\nu}(1+|\xi|)^{m_1+\delta_1\nu}(1+|\eta|)^{N-\nu}.\end{aligned}$$

If ν is large, we obtain a bound

$$|R_N(\eta,\xi)|\le C(1+|\xi|)^{m-(\rho-\delta)N}(1+|\eta|)^M\quad for\ |\eta|\le\frac{1}{2}|\xi|,$$

and

$$|R_N(\eta,\xi)|\le C_M(1+|\eta|)^{-M}\ for\ |\xi|\le 2|\eta|\le C_M\Big(1+\frac{1}{2}|\xi|\Big)^{-M/2}(1+|\eta|)^{-M/2}.$$

From these estimates, (3. 3) follows.

To prove (3. 4), write

$$D_x{}^{\beta}D^{\alpha}a(x,\xi)=\sum_{\alpha_1+\alpha_2=\alpha}C_{\alpha_1,\alpha_2}\iint(y-x)^{\alpha_1}(\xi-\xi_2)^{\beta}e^{i(y-x)(\xi-\xi_2)}D_{\xi_2}{}^{\alpha_2}b(\xi_2,y,\xi)\,dyd\xi_2.$$

Thus we need a bound

(3. 6) $$\Big|\iint y^{\alpha_1}\xi_2{}^{\gamma}e^{i(y-x)\cdot(\xi-\xi_2)}D_\xi{}^{\alpha_2}b(\xi_2,y,\xi)\,dyd\xi_2\Big|\le C(1+|\xi|)^{\mu}.$$

The left side is equal to (with $b_1=y^{\alpha_1}b$)

$$\begin{aligned}&\int\xi_2{}^{\gamma}e^{-ix(\xi-\xi_2)}D^{\alpha_2}{}_{\xi_1}\hat{b}_1(\xi_2,\xi-\xi_2,\ \xi_1)\,|_{\xi_1=\xi_2}d\xi_2\\&=\int(\xi+\eta)^{\gamma}e^{-ix\cdot\eta}D_{\xi_1}{}^{\alpha_2}\hat{b}_1(\xi+\eta,\eta,\xi_1)\,|_{\xi_1=\xi}dn.\end{aligned}$$

The integrand in this last expression is bounded in absoluste value by

$$C_\gamma|\xi+\eta|^{|\gamma|}(1+|\xi+\eta|)^{m_2+\delta_2\nu}(1+|\xi|)^{m-\rho|\alpha|+\delta_1\nu}(1+|\eta|)^{-\nu}$$

by (3. 5). From this, (3. 6) easily follows, and we obtain (3. 4).

Recall that if $a(x,\xi)\in S^m_{\rho,\delta}$, the single pseudodifferential operator $A=a(x,D)$ is given by

$$Au(x)=\frac{1}{(2\pi)^n}\int e^{ix\xi}a(x,\xi)\hat{u}(\xi)\,d\xi.$$

THEOREM 3. 3. *Let* $a(x,\xi)\in S^{m_1}_{\rho,\delta^1}$ *and* $b(\xi_2,x_1,\xi_1)\in S^{m_2,m_3}_{\rho,\delta_1,\delta_2}$. *Suppose* $b(\xi_2,x_1,\xi_1)$ *has compact support in* x_1*-components and* $0\le\delta<\rho\le 1$ *with* $\delta=\delta_1+\delta_2$. *Then* $c(x,D)=a(x,D)b(D,x_1,D)\in S^m_{\rho,\delta}$ $(m=m_1+m_2+m_3)$ *has the asymptotic expansion*

(3. 7) $$c(x,\xi)\sim\sum_{r\ge 0}c_r(x,\xi)$$

where

$$c_r(x,\xi)=\sum_{|\alpha+\beta|=r}\frac{1}{\alpha!\beta!}\partial_\xi{}^\alpha a(x,\xi)\partial_{\xi_2}{}^\beta D_x{}^{\alpha+\beta}b(\xi_2,x,\xi)\,|_{\xi_2=\xi}.$$

Proof. Using Fubini Theorem, for $u\in\mathcal{S}$ (with $B=b(D,x_1,D)$)

$$c(x,D)u(x)=\frac{1}{(2\pi)^n}\int e^{ix\cdot\xi_2}a(x,\xi_2)\widehat{Bu}(\xi_2)d\xi_2$$

$$=\frac{1}{(2\pi)^{2n}}\iiint e^{ix\cdot\xi_2+iy\cdot(\xi_1-\xi_2)}a(x,\xi_2)b(\xi_2,y,\xi_1)\hat{u}(\xi_1)dyd\xi_2d\xi_1$$

From this, we have

$$c(x,\xi_1)=\frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot(\xi_2-\xi_1)}a(x,\xi_2)b(\xi_2,y,\xi_1)dyd\xi_2$$

$$=\frac{1}{(2\pi)^n}\int e^{ix\cdot\eta}a(x,\xi_1+\eta)\hat{b}(\xi_1+\eta,\eta,\xi_1)d\eta.$$

By Taylor's formula, we have

$$a(x,\xi_1+\eta)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\eta^\alpha\partial_{\xi_1}{}^\alpha a(x,\xi_1)+r_N(a)(\eta,\xi_1),$$

$$\hat{b}(\xi_1+\eta,\eta,\xi_1)\sum_{|\beta|<N}\frac{1}{\beta!}\eta^\beta\partial_{\xi_2}{}^\beta\hat{b}(\xi_2,\eta,\xi_1)\,|_{\xi_2=\xi_1}+r_N(\hat{b})(\eta,\xi_1)$$

where

$$r_N(a)(\eta,\xi_1)=\frac{1}{(N-1)!}\int_0^1(1-t)^{N-1}\partial_t{}^N(x,\xi_1+t\eta)dt$$

$$r_N(\hat{b})(\eta,\xi_1)=\frac{1}{(N-1)!}\int_0^1(1-t)^{N-1}\partial_t{}^N\hat{b}(\xi_1+t\eta,\eta,\xi_1)dt.$$

Therefore

$$a(x,\xi_1+\eta)\hat{b}(\xi_1+\eta,\eta,\xi_1)$$
$$=\sum_{r<N}\Big(\sum_{|\alpha+\beta|=r}\frac{1}{\alpha!\beta!}\eta^{\alpha+\beta}\partial_{\xi_1}{}^\alpha a(x,\xi_1)\partial_{\xi_2}{}^\beta\hat{b}(\xi_2,\eta,\xi_1)\,|_{\xi_2=\xi_1}\Big)+r_N(c)(\eta,\xi_1).$$

Taking inverse Fourier transforms with respect to η yields

$$c(x,\xi_1)=\sum_{r<N}c_r(x,\xi_1)+\tilde{r}_N(c)(x,\xi_1)$$

where $\tilde{r}_N(c)(x,\xi_1)=\frac{1}{(2\pi)^n}\int e^{ix\cdot\xi_1}r_N(c)(\eta,\xi_1)d\eta$. Then clearly $c_r(x,\xi_1)\in S_{\rho,\delta}{}^{m-(\rho-\delta)|r|}$ with $m=m_1+m_2+m_3$. To prove $|r_N|\le C(1+|\xi_1|)^{m-(\rho-\delta)N}$ and $c(x,\xi_1)\in S^m_{\rho,\delta}$ it is only necessary to verify the following estimates:

$$(3.8)\qquad \Big|\int e^{ix\cdot\eta}\eta^\beta\partial_{\xi_2}{}^\beta\hat{b}(\xi_2,\eta,\xi_1)\,|_{\xi_2=\xi_1}\,r_N(a)(\eta,\xi_1)d\eta\Big|\le C_{N,\beta}(1+|\xi_1|)^{m-(\rho-\delta)N},$$

$$(3.9)\qquad \Big|\int e^{ix\cdot\eta}\eta^\alpha\partial_\xi{}^\alpha a(x,\xi_1)r_N(\hat{b})(\eta,\xi_1)d\eta\Big|\le c_{\alpha,N}(1+|\xi_1|)^{m-(\rho-\delta)N},$$

$$(3.10)\qquad \Big|\int e^{ix\cdot\eta}\eta^{\alpha+\beta}\partial_{\xi_1}{}^\alpha a(x,\xi_1)\partial_{\xi_2}{}^\beta\hat{b}(\xi_2,\eta,\xi_1)\,|_{\xi_2=\xi_1}d\eta\Big|\le c_{\alpha\beta}(1+|\xi_1|)^{m-(\rho-\delta)N}$$

for $N<|\alpha+\beta|<2N$,

$$(3.11)\qquad |D_x{}^\beta D_\xi{}^\alpha c(x,\xi)|\le c_{\alpha\beta}(1+|\xi|)^{\mu(\alpha,\beta)}$$

The proofs of (3.8), (3.9) and (3.11) are similar to the proof of Theorem 3.2. To prove (3.10), we obtain

$$\Big|\int e^{ix\cdot\eta}\eta^{\alpha+\beta}\partial_{\xi_1}{}^\alpha a(x,\xi_1)\partial_{\xi_2}{}^\beta\ \hat{b}(\xi_2,\eta,\xi_1)\,|_{\xi_2=\xi_1}d\eta\Big|$$
$$\le c_1\int|\eta|^{|\alpha+\beta|}(1+|\xi_1|)^{m-\rho|\alpha+\beta|+\delta\nu}(1+|\eta|)^{-\nu}d\eta$$

With sufficiently large ν, we obtain (3. 10). This completes the proof of Theorem.

References

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