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CONTINUITY OF HOMOMORPHISMS BETWEEN BANACH ALGEBRAS

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1. Introduction

The problems of the continuity of homomorphisms between Banach algebras have been studied widely for the last two decades to obtain various fruitful results, yet it is far from characterizing the class of Banach algebras for which each homomorphism from a member of the class into a Banach algebra is conitnuous. For commutative Banach algebras A and B a simple proof shows that every homomorphism θ from A into B is continuous provided that B is semi-simple, however, with a non semi-simple Banach algebra B examples of discontinuous homomorphisms from $C(K)$ into B have been constructed by Dales [6] and Esterle [7]. For non commutative Banach algebras the problems of automatic continuity of homomorphisms seem to be much more difficult. Many positive results and open questions related to this subject may be found in [1], [3], [5] and [8], in particular most recent development can be found in the Lecture Note which contains [1].

It is well-known that a *-isomorphism from a C^*-algebra into another C^*-algebra is an isometry, and an isomorphism of a Banach algebra into a C^*-algebra with self-adjoint range is continuous. But a *-isomorphism from a C^*-algebra into an involutive Banach algebra is norm increasing [9], and one can not expect each of such isomorphisms to be continuous. In this note we discuss an isomorphism from a commutative C^*-algebra into a commutative Banach algebra with dense range via separating space. It is shown that such an isomorphism $\theta : A \to B$ is conitnuous and maps A onto B if B is semi-simple, discontinuous if B is not semi-simple.

2. Separating Spaces

Let A and B be Banach spaces, and let $T : A \to B$ be a linear operator from A into B. The *separating space* $\mathfrak{S}(T)$ is defined by

$$\mathfrak{S}(T) = \{b \in B : \text{there exists a sequence } \{a_n\} \text{ in } A \text{ with } a_n \to 0 \text{ and } T(a_n) \to b\}.$$

LEMMA 1. *Let $\mathfrak{S}(T)$ be the separating space of a linear operator $T : A \to B$, then*
(i) *T is continuous if and only $\mathfrak{S}(T) = \{0\}$,*
(ii) *$\mathfrak{S}(T)$ is a closed linear subspace of B.*

Proof. (i) If $\mathfrak{S}(T) = \{0\}$, then clearly T is continuous by the closed graph theorem.

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(ii) Clearly $\mathfrak{S}(T)$ is a linear subspace of B. Let $b\in\overline{\mathfrak{S}(T)}$, the norm closure of $\mathfrak{S}(T)$. Then for each natural number n there is a $b_n\in\mathfrak{S}(T)$ with $\|b-b_n\|<\frac{1}{n}$. Choose a sequence $\{a_{ni}\}$ in A with $\lim_{i\to\infty} a_{ni}=0$ and $\lim_{i\to\infty}(a_{ni})=b_n$ for each n. Put $a_n=a_{nn}$, then we have $a_n\to 0$ and $T(a_n)\to b$. Thus $b\in\mathfrak{S}(T)$.

The *radical* of a Banach algebra A, denoted by $\mathrm{rad}(A)$, is defined by the intersection of the kernels of all irreducible representations of A, thus $\mathrm{rad}(A)$ is the intersection of the maximal modular left ideals of A [4]. If A is commutative, then $\mathrm{rad}(A)$ is the intersection of the kernels of all characters on A. A Banach algebra A is called *semi-simple* if $\mathrm{rad}(A)=\{0\}$.

LEMMA 2. *Let A and B be Banach algebras and let $\theta: A\to B$ be a homomorphism from A into B. Then the separating space $\mathfrak{S}(\theta)$ is a closed two sided ideal of $\overline{\theta(A)}$. Moreover, if A and B are both commutative then $\mathfrak{S}(\theta)\subset\mathrm{rad}(B)$.*

Proof. Let $b\in\mathfrak{S}(\theta)$. For each $y\in\theta(A)$ it is strightforward to show that yb and by belong to $\mathfrak{S}(\theta)$. Let $y\in\overline{\theta(A)}$. Choose a sequence $\{y_n\}$ in $\theta(A)$ with $y_n\to y$. Since $y_n b\in\mathfrak{S}(\theta)$, $y_n b\to yb$ and $\mathfrak{S}(\theta)$ is closed, we have $yb\in(\theta)$. Similarly $by\in\mathfrak{S}(\theta)$. Hence $\mathfrak{S}(\theta)$ is a closed two sided ideal of $\overline{\theta(A)}$. Suppose that both A and B are commutative and let $b\in\mathfrak{S}(\theta)$. Let $\{a_n\}$ be a sequence in A such that $a_n\to 0$ and $\theta(a_n)\to b$. Since each character on a commutative Banach algebra is continuous [4] and for each charactor Ψ on B $\Psi\circ\theta$ is a character on A, we have

$$\begin{aligned}\Psi(b) &= \Psi(\lim\theta(a_n))\\ &= \lim\Psi\circ\theta(a_n)\\ &= \Psi\circ\theta(0)=0.\end{aligned}$$

Therefore, $b\in\mathrm{rad}(B)$ and we have $\mathfrak{S}(\theta)\subset\mathrm{rad}(B)$.

The following well-known proposition is an immediate consequence of the above lemma.

PROPOSITION 1. *If $\theta: A\to B$ is a homomorphism of a commutative Banach algebra A into a semi-simple commutative Banach algebra B, then θ is continuous.*

Proof. $\mathfrak{S}(\theta)\subset\mathrm{rad}(B)=\{0\}$ and θ is continuous.

For non commutative Banach algebras A and B it is not known whether or not a homomorphism $\theta: A\to B$ for which $\overline{\theta(A)}$ is semi-simple is automatically continuous. The following is a partial answer to the above open problem proved by Aupetit [2].

PROPOSITION 2. *Let $\theta: A\to B$ be a homomorphism from a Banach algebra into a semi-simple Banach algebra such that the range $\theta(A)$ is dense in B with finite or countable codimension. Then θ is continuous and maps A onto B.*

3. Isomorphisms with dense range

THEOREM. *Let A be a commutative C^*-algebra and B a commutative Banach alge-*

bra. If $\theta : A \to B$ is an isomorphism with dense range, then $B=\theta(A)\oplus \mathcal{S}(\theta)$ and $\mathcal{S}(\theta)=rad(B)$. Moreover, if B is semi-simple then θ is continuous and maps A onto B, if B is not semi-simple then θ is discontinuous.

Proof. First we will show that $\|\theta(x)\|\geq\|x\|$ for each $x\in A$. By adjunction of an identity we may assume that A and B are both unital, and let $\Omega(A)$ and $\Omega(B)$ debote the spectrums of A and B respectively.

Clearly the transpose ${}^t\theta$ of θ maps $\Omega(B)$ into $\Omega(A)$. Let

$$U=\{a'\in A' : |a'(x_i)|<\varepsilon,\ x_i\in A,\ i=1,2,\cdots,n\}$$

for an $\varepsilon>0$ and a positive integer n. Put $y_i=\theta(x_i)$, $i=1,2,\cdots,n$, and let

$$V=\{b'\in B' : |b'(y_i)|<\varepsilon,\ i=1,2,\cdots,n\}.$$

Then we have $({}^t\theta)^{-1}(U)=V$, thus the restriction of ${}^t\theta$ on the spectrum $\Omega(B)$ is continuous. Suppose that ${}^t\theta(\Omega(B))\neq\Omega(A)$. Since ${}^t\theta(\Omega(B))$ is a compact subset of $\Omega(A)$, for each ω_0 in $\Omega(A)$ which does not belong to ${}^t\theta(\Omega(B))$ there exists an open neighborhood W of ω_0 with $\overline{W}\cap{}^t\theta(\Omega(B))=\phi$, so that we may choose a continuous function f on $\Omega(A)$ such that

$$f(\omega)=1 \text{ on } {}^t\theta(\Omega(B)) \text{ and } f(\omega)=0 \text{ on } W.$$

Since the Gelfand transformation $F: A\to C(\Omega(A))$, $F(a)=\hat{a}$, $a\in A$, is an isometric isomorphism of A onto $C(\Omega(A))$, there exists an a in A with $\hat{a}=f$. Similarly we may choose an element b in A such that

$$\hat{b}(\omega_0)=1 \text{ and } \hat{b}(\omega)=0 \text{ on } \Omega(A)\sim W.$$

Then we have $\hat{a}\hat{b}=0$ and hence $\theta(a)\theta(b)=0$. Also, by the choice of a, $\omega(\theta(a))=1$ for each ω in $\Omega(B)$ so that $\theta(a)$ does not belong to any maximal ideal of B. Thus $\theta(a)$ is invertible and hence $\theta(b)=0$. But $\hat{b}(\omega_0)=1$ and $\theta(b)\neq0$. This contradiction shows that $\Omega(A)={}^t\theta(\Omega(B))$.

Now, for each x in A we have

$$\begin{aligned}\|\theta(x)\|\geq\|\theta(x)\|_\infty&=\sup\{|\omega(\theta(x))| : \omega\in\Omega(B)\}\\&=\sup\{|({}^t\theta\omega)(x)| : \omega\in\Omega(B)\}\\&=\sup\{|\omega'(x)| : \omega'\in\Omega(A)\}\\&=\|\hat{x}\|_\infty=\|x\|.\end{aligned}$$

Suppose that x is a non zero element of A, then for each positive integer n

$$\|\theta(x^n)\|\geq\|x^n\|=\|\hat{x}^n\|_\infty=\|\hat{x}\|_\infty{}^n>0.$$

Thus the spectral radius of $\theta(x)$ is positive and $\theta(x)\notin\mathrm{rad}(B)$. Therefore we have

$$\theta(A)\cap\mathrm{rad}(B)=\{0\}.$$

Let $y\in B=\overline{\theta(A)}$ and let $\{x_n\}$ be a sequence in A with $\theta(x_n)\to y$. Since $\|x_n-x_m\|\leq\|\theta(x_n-x_m)\|$there exists an x_0 in A with $x_n\to x_0$, hence we have

$$\lim_{n\to\infty}\theta(x_n-x_0)=\lim_{n\to\infty}\theta(x_n)-\theta(x_0)$$

$$= y - \theta(x_0) \in \mathcal{S}(\theta) \subset \mathrm{rad}(B).$$

Since we can write

$$y = \theta(x_0) + (y - \theta(x_0))$$

we have

$$B = \overline{\theta(A)} = \theta(A) \oplus \mathcal{S}(\theta) = \theta(A) \oplus \mathrm{rad}(A).$$

The last equality shows that $\mathcal{S}(\theta) = \mathrm{rad}(B)$.

If B is semi-simple, then $\mathcal{S}(\theta) = \{0\}$. Thus θ is continuous and $\theta(A) = B$. If B is not semi-simple, then $\mathcal{S}(\theta) \neq \{0\}$ and hence θ is discontinuous.

COROLLARY. *If θ is a *-isomorphism from a commutative C^*-algebra A into a commutative involutive Banach algebra B, then θ is continuous if and only if $\overline{\theta(A)}$, the norm closure of the range of θ, is semi-simple. In this case, θ has the closed range.*

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