Bull. Korean Math. Soc.
Vol. 20, No. 1, 1983

DIRECT PRODUCT AND DIRECT SUM OF A LOCAL RING

Dedicated to the memory of Professor Dock Sang Rim

BYOUNG-SONG CHWE

A ring A, with the identity elements, is called a right Steinitz ring in [2] and [4] when the set R of non-units forms an ideal and left vanishing (or right T-nilpotent), in the sense that for any infinite sequence $\{x_i\}$ of the elements of R, there is an integer m such that

$$x_m \cdot x_{m-1} \cdots\cdots x_1 = 0.$$

This paper is an attempt to try to find a relation between a ring and its direct product in the term of Steinitz ring.

Let A be a local ring in the sense that the set R of nonunits forms an ideal. Let ΠA be the set of all maps from the set of natural integers, N, to A, and ΣA be the subset of ΠA consists of maps which take non-zero images only for finitely many elements of N. Take ΠA and ΣA as right A-module as usual and ΣA is a direct sum and ΠA is direct product of A. We want to prove the following two theorems:

THEOREM 1. *If ΣA is a direct summand of ΠA, then R is nilpotent.*

THEOREM 2. *R is left vanishing and finitely generated as right A module, then ΣA is a direct summand of ΠA.*

LEMMA 1. *Suppose ΣA is a direct summand of ΠA, then for any $f \in \Pi A$, there is an integer n satisfying following conditions. If $fa \in \Sigma A$ for some $a \in A$, then $f(i)a = 0$ for all $i > n$.*

Proof. Let ϕ be the A-homomorphism of ΠA onto ΣA leaving the elements of ΣA left fixed. If $fa \in \Sigma A$, then $\phi(fa) = fa$, and $\phi(fa) = \phi(f) \cdot a$. Let n be such that $\phi(f)(i) = 0$ for all $i > n$. Then $f(i)a = \phi(f)(i) \cdot a = 0$ for all $i > n$.

LEMMA 2. *Let $\{x_1, x_2, \ldots\ldots\}$ be a sequence of elements of R, and $\{g_1, g_2 \ldots\ldots\}$ be a sequence of elements of ΠA, defined by*

$$g_1 = \{1, x_1, x_2 \cdot x_1, x_3 \cdot x_2 \cdot x_1, \ldots\ldots\},$$
$$g_2 = \{0, 1, x_2, x_3 \cdot x_2, \ldots\ldots \qquad \}.$$

In general, $g_i(j) = 0$ *if* $j < i$,
$g_i(i) = 1$,

and

$$g_i(j) = x_{j-1} \cdot x_{j-2} \cdots\cdots x_i \text{ if } j > i.$$

Let V be free sub-module of πA spanned by $\{g_i\}$. Then if $V/\sum A$ is free-module, then $V=\sum A$, and R is left vanishing.

Proof. Let ϕ be the cannonical map of V to $V/\sum A$, and f_i be the element of $\sum A$ such that

$$f_i(j)=0 \text{ if } i\neq j$$
$$f_i(i)=1,$$

for each $i\in N$.

Since $g_i=f_i+g_{i+1}\cdot x_i$,

$$\phi(g_i)=\phi(g_{i+1})\cdot x_i.$$

Since $\{g_i\}$ spans V, and R forms an ideal, it is clear that there is a basis of $\phi(V)$ consisto of one of $\phi(g_i)$, if $\phi(V)\neq\{0\}$. Say $\{\phi(g_i)\}$ is a basis, then

$$\phi(g_i)=\phi(g_{i+1})x_i,$$

and $\phi(g_{i+1})=\phi(g_i)y$ for some $y\in A$, and

$$\phi(g_i)=\phi(g_i)yx_i.$$

Hence $yx_i=1$, this is a contradiction to $x_i\in R$. Hence $\phi(V)=\{0\}$. Therefore each g_i belongs to $\sum A$, and there is m such that

$$x_m\cdot x_{m-1}\dots x_i=0.$$

Hence R is left vanishing.

LEMMA 3. (from [4]). *Let A be a Steinitz ring with the radical R. Let $I_i=\{x\in A \mid R^i\cdot x=0\}$, for each $i\in N$. If R is not $\{0\}$ nor nilpotent then*

$$I_2\subset I_2\subset I_3\subset\dots\dots$$

is a strictly ascending infinite chain of two sided ideals of A.

Proof. Suppose $I_1=\{0\}$, then $R\cdot x=0$ implies $x=0$, hence there is a choice function f of $R-\{0\}$ to R such that $f(x)x\neq 0$. Define a sequence of elements of R by

$$x_1=x$$
$$x_2=f(x_1)$$
$$x_3=f(x_2\cdot x_1)$$

In general

$$x_i=f(x_{i-1}\cdot x_{i-2}\dots\dots x_1), \text{ for } i\in N.$$

Then there is no n such that

$$x_n\cdot x_{n-1}\dots x_1=0$$

this is contradicted that R is left vanishing. Therefore $I_1\neq\{0\}$, and also $I_1\neq R$ because $RR\neq\{0\}$.

Consider A/I_1, then its radical is not zero nor nilpotent, because $R^i\subset I_1$ implies $R\cdot R^i=R^{i+1}=\{0\}$. Since

$I_2=\{x\in A\mid R^2x=0\}=\{x\in A\mid R\cdot x\subset I_1\}$, by applying above argument,

$$I_1\subset I_2 \text{ and } I_1\neq I_2, \text{ and } I_2\neq R.$$

By repeating this process we get strictly ascending chains of two sided ideals. Actually ascending chain condition for two sided ideals implies that R is nolpotent when A is a Steinitz ring.

Proof of Theorem 1. Let $\{x_i\}$ and $\{g_i\}$ and V be as same in Lemma 2. If $\sum A$ is a direct summand of πA, then it is also direct summand of V, hence $V/\sum A$ is also a direct summand of V, and V is a free module. Therefore $V/\sum A$ is free by Kaplansky's theorem in [3], which says that projective module is free when the ring is a local ring. Hence from Lemma 2, A is a Steinitz's ring. Moreover if R is not nilpotent, then we have strictly ascending chain of ideals $I_1 \subset I_2$......as in Lemma 3. Let $\{a_1, a_2, \ldots\}$ be a sequence of element of R such that $a_i \in I_{i+1} - I_i$ for each $i \in N$. And $\{r_1, r_2, \ldots\}$ be also a sequence of the element of R such that $r_i \in R^i$ and $r_i a_i \neq 0$ for each i. Such $\{r_i\}$ exists because $a_i \in I_{i+1} - I_i$ implies $R^{i+1} a_i = \{0\}$ and $R^i a_i \neq \{0\}$. Let f be an element of πA such that $f(i) = r_i$ for $i \in N$, then $f a_i \in \sum A$ and $f(i) a_i \neq 0$ for each i. This is a contradiction to Lemma 1. Therefore if $\sum A$ is a direct summand of πA, A is a Steinitz ring such that the radical is nilpotent.

Proof of Theorem 2. Suppose the set of non-units, R is left vanishing and finitely generated as a right ideal of A. In [5], as a result of [1], it was proved that, if A is a right Steinitz ring, then the maximals ideal R is finitely generated as a right A-module if and only if any direct product of A is a free right-module. Therefore any direct product of A is free if A is Steinitz and R is finitely generated. Now, since πA is free and $\{f_i\}$ defined as in Lemma, a basis of $\sum A$, is linearly independent set, hence there is a basis of πA containing $\{f_i\}$, from [2]. Therefore $\sum A$ is a direct summand of πA.

It is left to study whether it is necessary for R to be finitely generated in order to make $\sum A$ a direct summand of πA.

References

1. Chase, Stephen V., *Direct products of modules*, Trans. Am. Math. Soc. **97** (1960), pp. 457-473.
2. B.S. Chwe and J. Neggers, *On the extension of linearly independent subset of free modules to a basis*, Proc. Amer. Math. Soc. **24** (1970), 466-470.
3. I. Kaplansky, *Projective module*, Amer. Math. **68** (1958), 372-377.
4. B.S. Chwe and J. Neggers, *Local ring with left vanishing radical*, J. London Math. Soc. (2), **4** (1971), 374-378.
5. B.S. Chwe and W.H. Rant, *A characterization of the minimum condition for a local ring and certain perfect ring*, to appear.

Department of Mathematics, University of Alabama, University, AL 35486, U.S.A