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ON THE REIDEMEISTER NUMBERS AND NIELSEN NUMBERS OF THE (EVENTUALLY ABELIAN) FIBER-PRESERVING MAPS

Dedicated to the memory of Professor Dock Sang Rim

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1. Introduction.

Let $\mathcal{F}=(E, p, B)$ be an orientable Hurewicz fibering with a regular lifting function λ, where E, B, and $p^{-1}(b)$, $b\in B$, are compact, connected, metric ANR's. A fibering is orientable if the translation map $\tau : p^{-1}(b)\to p^{-1}(b)$, defined by $\tau(e)=\lambda(e, w)$ (1) for every loop w at b in B, is homotopic to the identity. If $f : E\to E$ is a fiber-preserving map then f induces a continuous map $\bar{f} : B\to B$ such that $pf=\bar{f}p$ and a continuous map $f_b : p^{-1}(b)\to p^{-1}(b)$ for each $b\in B$ defined by $f_b(e)=\lambda(f(e), w)$ (1), where w is a path from $\bar{f}(b)$ to b in B. The triple $(f, \bar{f}, f_b) : \mathcal{F}\to\mathcal{F}$ is called a fiber-preserving map triple. It is known that the Nielsen number, $N(f_b)$, of the map f_b is independent of the choice of paths w from $\bar{f}(b)$ to b and points b in B([2] and [7]). It is well known that the Lefschetz numbers of the maps involved satisfy the relation $L(f)=L(\bar{f})\cdot L(f_b)$, $b\in B$. However the corresponding result for Nielsen numbers is false as shown in [4] and [15]. If the spaces involved in $\mathcal{F}$ satisfy the Jiang condition, that is, the Jiang subgroup $J(X)$ is equal to the fundamental group $\pi_1(X, x_0)$, hence the fundamental groups are abelian, then there is a complete solution to the problem of product relation between Nielsen numbers of a fiber-preserving map tirple, [15] and [16]. That is, if $L(f)\neq 0$ then there is an invariant $P(f)$ such that $N(f)\cdot P(f)=N(\bar{f})\cdot N(f_b)$ for a locally trivial fibering $\mathcal{F}$ and a fiber-preserving map triple $(f, \bar{f}, f_b)$: $\mathcal{F}\to\mathcal{F}$, ([15]. If the fundamental groups of the spaces involved are not abelian, then there are only partial solutions to the product theorem. These can be found in [2], [4], [6], [8], [14] and [16]. In this paper, we assume that a fiber-preserving map $f : E\to E$ has the non-zero Lefschetz number. Therefore, there is an essential fixed point class of f.

We choose a point $x_0\in E$ in an essential fixed point class of f and $p(x_0)\in B$ as the base points of E and B respectively. Then we will omit the base points in the sequel for the fundamental groups since the spaces are all path connected.

The purpose of this paper is to prove the above theorem for a Hurewicz fibering and to eliminate the Jiang conditon from the above theorem as much as possible. Now the new theorem reads that for an orientable Hurewicz fibering $\mathcal{F}=(E, p, B)$ and a fiber-preserving map triple $(f, \bar{f}, f_b) : \mathcal{F}\to\mathcal{F}$, if $\pi_1(E)$ is abelian and $(Lf)\neq 0$, and if the fibers satisfy the Jiang condition, then there is an invariant P depending on f such that $N(f)\cdot P=N(\bar{f})\cdot N(f_b)$. In this case, the fundamental groups are abelian. Therefore, our theorem generalizes theorems in [13] and [15].

We also prove a similar theorem for an eventually abelian fiber-preserving map. In [12, p. 61], Kiang defines an eventually abelian map to be a map $f: E \to E$ such that the image of the induced homomrphism f^n for some positive integer n becomes an abelian subgroup of $\pi_1(E)$, where $f_\#{}^n = f\ldots\ldots f$, n times, (see alse [9]).

Kiang [12 : p. 62] shows that if $f: E \to E$ is eventually abelian, then the Reidemeister classes of f has one to one conespondence with $H_1(E)/(1-f_{1*})(H_1(E))$, i.e., $R(f) = \text{Ord}\ (H_1(E)/(1-f_{1*})(H_1(E))$, where $H_1(E)$ is the first integral homology group and 1 indicates the identity homomorphism of $H_1(E)$.

We note that, in general, $R(f) \geq \text{Ord}\ (H_1(E)/(1-f_{1*})(H_1(E))$. We also note that if there are two positive integers m, n such that $f_\#{}^m\ (\pi_1(E)) \subset f_\#{}^n(J(f))$, then f is eventually abelian and hense $R(f) = \text{Ord}\ (H_1(E)/(1-f_{1*})(H_1(E))$, [11], where $J(f)$ is the Jiang subgroup of the map f. In general the Nielsen number $N(f)$ is less than or equal to the Reidemeister number $R(f)$. However, if $J(f) = \pi_1(E)$, then $N(f) = R(f)$, [1].

You [16] made a remark that the product relation $N(f)P(f) = N(\bar{f}) \cdot N(f_b)$ for a fiber-preserving map triple would hold if both f and f_b are eventually abelian. We prove that if the inclusion map $i: p^{-1}(b) \subset E$ induces a monomrphism $i_\#: \pi_1(p^{-1}(b)) \to \pi_1(E)$ and if f is eventually abelian, then both $\bar{f}$ and f_b are also eventually abelian. This fact then gives a very simple proof of the product theorem for the Nielsen numbers of an eventually abelian fiber-preserving map triple.

In section 2, we study on algebraic notions of Reidemeister classes of the group, and present our proofs of the theorems in section 3 for the case when the fundamental groups are abelian, and in section 4 for the case when f is eventually abelian. In section 5, we examine a general case when the spaces involved are aspherical.

We are heavily depended on [1], [2], [6], and [9], so that interested readers are referred to these articles for more information.

2. On the Reidemeister homomorphisms

Let $h: G \to G$ be a homomorphism on a group G into itself. Two elements α and β are said to be h-equivalent if there exists an element $\delta \in G$ such that $\alpha = \delta\beta h(\delta^{-1})$. This is an equivalence relation on G and divides G into equivalence classes $G' = G/''\sim'' = \{[\alpha]\}$, the cardinality of this set is called the Reidemeister number of h on G, and it is denoted by $R(h)$. If G is an abelian group then we define the addition in G' by $[\alpha]+[\beta] = [\alpha+\beta]$ for elements $[\alpha]$ and $[\beta]$ in G'.

It is not so hard to show that this is a well defined operation and G' becomes an abelian group, in fact, G' is isomorphic to the cokernel of id. $-h: G \to G$.

Let $h': H \to H$ be an another homomorphism on a group H. If $i: H \to G$ is a homomorphism such that $ih' = hi$ then i induces $i^\#: H' \to G'$ defined by $i^\#([\alpha]) = [i\alpha]$. It is easy to see that if α and β are h'-equivalent then $i\alpha$ and $i\beta$ are h-equivalent. Thus $i^\#$ is well defined and we have the following lemmas.

LLMMA 2.1. *If H and G are abelian groups then $i^\#: H' \to G'$ is a homomorphism.*

LEMMA 2.2. *If $h: G \to G$, $h': H \to H$ and $i: H \to G$ are homomorphisms of abelian* *groups such that* $hi=ih'$, *and if there exists a homomorphism* $j: G \to H$ *such that* $ji = id_H$ *and* $h'j=jh$, *then* $i^{\#}: H' \to G'$ *is a monomorphism.*

Proof. If $[\alpha], [\beta]$ are elements in H' such that $i^{\#}([\alpha]) = i^{\#}([\beta])$ then $i\alpha$ and $i\beta$ are h-equivalent, i. e., there exists an element $\delta \in G$ such that $i\alpha = \delta + i\beta - h(\delta)$. This implies that $ji\alpha = i\delta + ji\beta - ih(\delta)$ and $\alpha = i\delta + \beta - h'j(\delta)$, and $j(\delta) \in H$. Therefore α and β are h-equivalent. (See Proposition (2.1) in [14])

LEMMA 2.3. *If* $h: G \to G$, $h': H \to H$ *are homomorphisms of abelian groups and if* $i: H \to G$ *is an isomorphism such that* $hi=ih'$ *then* $i^{\#}: H' \to G'$ *is also an isomorphism. This follows from Lemma* 2.2.

LEMMA 2.4. *If* H *is a subgroup of an abelian group* G *and* h' *is the restriction to* H *of a homomorphism* $h: G \to G$ *and* $i: H \subset G$ *is the inclusion satisfying* $hi = id_H$ *and* $h(G) \subset H$, *then* $i^{\#}: H' \to G'$ *is a monomorphism. This also follows from Lemma* 2.2.

Now we relate the essential Nielsen fixed point classes and Reidemeister classes. Let $f: X \to X$ *be a continuous map on a compact, connected, metric ANR* X *into itself, such that* $L(f) \neq 0$. *This implies that there exists at least one essential fixed point class. Let* $\Phi(f) = \{x \in X \mid f(x) = x\}$ *and denote the essential Nielsen fixed point classes of* f *by* $\Phi'(f)$. *Let* $f_{\#}: \pi_1(X) \to \pi_1(X)$ *be the induced map and* $\pi_1'(X)$ *denote the corresponding set of Reidemeister equivalence classes of* $f_{\#}$. *We need the following proposition from* [1 : p. 104].

PROPOSITION 2.5. *There is an injective map* $\phi: \Phi'(f) \to \pi_1'(X)$.

3. On the fiber-preserving maps

Let $\mathfrak{F} = (E, p, B)$ be an orientable Hurewicz fibering where E, B, and $p'(b), b \in B$, are compact, connected, metric ANR's, and $(f, \bar{f}, f_b): \mathfrak{F} \to \mathfrak{F}$ be a fiber-preserving map triple defined in the introduction.

We assume $L(f) \neq 0$ and all the fundamental groups of spaces involved are abelian. The fiber homotopy exact sequence induces the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \to & \pi_2(B) & \xrightarrow{\partial} & \pi_1(p^{-1}(b)) & \xrightarrow{i_{\#}} & \pi_1(E) & \xrightarrow{P_{\#}} & \pi_1(B) & \to 0 \\
& \downarrow \bar{f}_{\#} & & \downarrow f_{b\#} & & \downarrow f_{\#} & & \downarrow \bar{f}_{\#} & \\
\cdots \to & \pi_2(B) & \xrightarrow{\partial} & \pi_1(p^{-1}(b)) & \xrightarrow{i_{\#}} & \pi_1(E) & \xrightarrow{P_{\#}} & \pi_1(B) & \to 0
\end{array}$$

Without loss of generality we may assume $b \in \Phi(\bar{f})$ and then $f_b = f | p^{-1}(b)$. From Lemma (2.1) the homomorphism $i_{\#}: \pi_1(p^{-1}(b)) \to \pi_1(E)$ induces a homomorphism $i^{\#}: \pi_1'(p^{-1}(b)) \to \pi_1'(E)$. Denote $P(f) =$ order of ker $i^{\#}$.

We state here two propositions which are due to Brown and Fadell.

PROPOSITION 3.1. (Brown [2]). *Let* $\mathfrak{F} = (E, p, B)$ *be a locally trivial fibering with* B *and fibrers finite polyhedra. Let* $(f, \bar{f}, f_b): \mathfrak{F} \to \mathfrak{F}$ *be a fiber-preserving map triple as before. Let* F_E *be a fixed point class of* f *and let* F_B *be the fixed point class of* $\bar{f}$

containing $p(F_E)$. *For each* $b \in F_B$, *let* $F_{y_1}, F_{y_2}, \dots, F_{y_k}$ *be the fixed point classes for* $f_b = f|_{p^{-1}(b)}$ *contained in* $(F_E \cap p^{-1}(b))$.

Then (1) $i(F_E) = i(F_B) \cdot \left(\sum_{n=1}^{k} i(F_{y_n})\right)$,

(2) *if* $p(F_E) \subsetneqq F_B$, *then* $i(F_{y_n}) = 0, 1 \leq n \leq k$, *and so* $i(F_E) = 0$, *where* $i(\ )$ *denotes the fixed point index.*

This is a generalization of Brown's results in [2 : p. 92-93] because his proof requires only that the base and all fibers are finite polyhedra, and because $F_E \cap p^{-1}(b)$ contains k distinct fixed point classes of the map f_b.

PROPOSITION 3.2. (Fadell, [6], [7]). *Let* $\mathfrak{F} = (E, p, B)$ *be an orientable Hurewicz fibering such that the spaces involved are compact, connected, metric ANR's, and let* $(f, \bar{f}, f_b) : \mathfrak{F} \to \mathfrak{F}$ *be a fiber-preserving map triple. Then there exists a locally trivial fibering* $\mathfrak{F}' = (E', p', B')$ *with both the base space and the fiber finite polyhedra and with a fiber-preserving map triple* $(f', \bar{f}', f_b') : \mathfrak{F} \to \mathfrak{F}'$ *such that* $N(\bar{f}) = N(\bar{f}')$, $N(f) = N(f')$ *and* $N(f_b) = N(f_b')$. *Furthermore, if* $\pi_1(E)$ *is abelian then* $\pi_1(E')$ *is also abelian and if fibers of* $\mathfrak{F}$ *satisfy the Jiang condition then fibers of* $\mathfrak{F}'$ *also satisfy the Jiang condition.*

The last part of the proposition follows from the construction of the locally trivial fibering $\mathfrak{F}$ because E' is homotopic to $E \times T^n$ and the fiber of $\mathfrak{F}'$ is homotopic to $p^{-1}(b) \times T^n$, where T^n is the n-dimensional torus.

Now we state our main theorem in this section.

THEOREM 3.3. *Let* $\mathfrak{F} = (E, p, B)$ *be an orientable Hurewicz fibering where* E, B, *and* $p^{-1}(b)$, $b \in B$, *are compact, connected, metric ANR's. Let* $(f, \bar{f}, f_b) : \mathfrak{F} \to \mathfrak{F}$ *be a fiber-preserving map triple such that* $L(f) \neq 0$. *If* $\pi_1(E)$ *is abelian and if fibers satisfy the Jiang condition the then there is a number* P *such that*

$$N(f) \cdot P(f) = N(\bar{f}) \cdot N(f_b), \quad b \in B.$$

We note that the number P *should be same as* $P(f) =$ *oder of Ker* $i^{\#}$, *defined in the begining of the section. However, we do not claim this fact here.*

Proof. We consider the locally trivial fibering $F' = (p', E', B')$ with the base B' and the fiber finite polyhedra, which corresponds to the fibering $F = (p, E, B)$ according to Proposition (3.2).

Then $\pi_1(E')$ is abelian and the fiber satisfies the Jiang condition and there exists at least one essential Nielsen fixed point class of f'. This also follows from the construction of the locally trivial fibering in Proposition (3.2). For each element F of the essential Nielsen fixed point classes of f', $\Phi'(f')$. if $p'^{-1}(b) \cap F \neq \phi$ we map the Nielsen classes of f_b' in $p'^{-1}(b) \cap F$ to F. This induces a map from $\Phi'(f_b')$ to $\Phi'(f')$. From Proposition (2.5) this map decomposes

$$\Phi'(f_b') \overset{1:1}{\rightarrow} \pi_1'(p'^{-1}(b)) \overset{i\#}{\rightarrow} \pi_1'(E') \overset{1:1}{\leftarrow} \Phi'(f').$$

Since the fiber satisfies the Jiang condition, $\Phi'(f_b') \to \pi_1'(p'^{-1}(b))$ is one to one and onto. Thus the number of Nielsen classes of f_b' in $p'^{-1}(b) \cap F$ which mapped to F is exactly the order of $\ker(i'^{\#})$. We see that this number is independent of the choice of the class $F \in \Phi'(f')$ since $i'^{\#}$ is a homomorphism. Since there exists at least one essential Nielsen fixed point class of f' and the fiber is a Jiang space, each essential Nielsen class of f' contains exactly $P(f')$ essential Nielsen classes of f_b', when $F \cap p'^{-1}(b) \neq \phi$, by the equation (1) of Proposition (3.1). It also follows from the same proposition that for each essential Nielsen fixed point class F of f', $p(F)$ in B is an essential Nielsen fixed point class of $\bar{f}'$ in B'. Thus over each essential Nielsen fixed point class of $\bar{f}'$ in B' there lies precisely $N(f_b')/P(f')$ essential Nielsen fixed point classes of f', so since $\bar{f}'$ has $N(\bar{f}')$ essential classes

$N(f')=N(\bar{f}')\cdot N(f_b')/P(f')$. We apply Proposition (3.2) to obtain the formula $N(f)\cdot P=N(\bar{f})N(f_b)$, $b\in B$, where $P=P(f')$.

The above theorem together with Lemmas (2.2), (2.3), and (2.4) imply the following:

COROLLARY 3.4. *We have the product relation* $N(f)=N(\bar{f})\cdot N(f_b), b\in B$, *in each of the following cases*:

(1) $i_{\#}:\pi_1(p^{-1}(b))\rightarrow\pi_1(E)$ *is monic and* $f_{\#}(\pi_1(E))\subset i_{\#}(\pi_1(p^{-1}(b))$ *and* $f_{\#}|_{i\#(\pi_1(p^{-1}(b)))}$ $=$ *identity.*

(2) $\pi_1(B)=0=\pi_2(B)$

(3) $0\rightarrow\pi_1(p^{-1}(b))\rightarrow^{p\#}\pi_1(E)\rightarrow^{p\#}\pi_1(B)\rightarrow 0$ *splits and splitting is natural with respect to the homomorphism* $f_{\#}$.

(4) $\pi_1(p^{-1}(b))=0$.

We note that the example in [4] shows that the hypothesis of $i_{\#}$ a monomorphism in the corollary can not be omitted.

Now we give an example of a calculation of the Nielsen number of a map which would not be possible without Theorem (3.3).

EXAMPLE 3.5. We consider the principal circle bundles over the 2-dimensional real projective space $RP(2)$. The even dimensional real projective spaces are known to not satisfy the Jiang condition. These bundles are orientable bundles and classified by $[RP(2), CP(\infty)]=H^2(RP(2), Z)=Z_2$. So that we have total spaces, $RP(2)\times S^1$ and non-trivial one E. The product theorem for the Nielsen numbers of a fiber-preserving map on the product spaces is covered in [3]. It is not hard to see that the fundamental group of E is abelian and in fact it is isomorphic to Z. Thus the fiber homotopy exact sequence becomes

$$0\rightarrow Z\overset{2}{\rightarrow}Z\rightarrow Z_2\rightarrow 0.$$

Now we construct a fiber-preserving map. Let us consider $RP(2)$ as a disk D^2 with antipodal points on the boundary $\dot{D}$ identified. Let d be an odd integer with $|d|>1$ and $f: D^2\rightarrow D^2$ be the map given by $f(re^{i\theta})=re^{id\theta}$. The fixed point set is $\frac{|1-d|}{2}$ lines passing through the center of D^2. Since f commutes with antipodal map on the boundary f induces $\bar{f}: RP(2)\rightarrow RP(2)$ with $N(\bar{f})=1$. Since the composition of maps $\bar{f}$ and the classifying map is homotopic to a classifying map, $\bar{f}$ can be lifted to a fiber-preserving map $f: E\rightarrow E$. Then f_b has degree d and $N(f_b)=|1-d|$ from [1]. From our Theorem (3.3) and [13] we have

$$N(f)=\frac{N(f_b)}{P}=\frac{|1-d|}{\frac{2d}{[2,d]}}=|1-d|, \text{ where}$$

$[2,d]$ is the least common multiple of 2 and d.

Finally, we would like to make a reamrk when the fundamental groups of the spaces in $\mathfrak{F}=(E,p,B)$ are not necessarily abelian.

Let G be a group and $g:G\to G$ be a homomorphism as in section 2. Then the Reidemeister classes G' does not have a group structure. Even if the conditions in lemmas in section 2 are satisfied, the cardinality of the inverse images of elements in G' under the map $i^\#$ are not the same, where $i^\#:H'\to G'$ induced by an homomorphism $i:H\to G$. Thus it would be interesting to find out under what conditions the cardinality of the inverse images of elements in G' under the map $i^\#$ are equal to 0 or the same because if the cardinality of $i^{\#-1}(\alpha)$, where $i^\#:\pi_1'(p^{-1}(b))\to\pi_1'(E)$, $\alpha\in\pi_1'(E)$, is 0 or a constant P then the statement

$$N(f)\cdot P=N(\bar{f})\cdot N(f_b),\ b\in B, \text{ is still true.}$$

4. Eventually abelian maps

In this section, we consider the case when the fiber-preserving map f is eventually abelian. First, we prove the following lemna.

LEMMA 4.1. *Let $F=(p,E,B)$ be an orientable Hurewicz fibering and $(f,\bar{f},f_b)$ be a fiber-preserving map triple.*

If the inclusion map $i:p^{-1}(b)\subsetneq E$ induces a monomorphism $i_\#:\pi_1(p^{-1}(b))\to\pi_1(E)$ and if f is eventually abelian, then $\bar{f}$ and f_b are eventually abelian.

Proof. The fiber preserving map $f:E\to E$ induces the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \to & \pi_1(p^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{P_\#} & \pi_1(B) & \to & 0 \\ & & \downarrow f_{b\#} & & \downarrow f_\# & & \downarrow \bar{f}_\# & & \\ 0 & \to & \pi_1(p^{-1}(b)) & \xrightarrow{i_\#} & \pi_1(E) & \xrightarrow{P_\#} & \pi_1(B) & \to & 0. \end{array}$$

First we show that $\bar{f}$ is eventually abelian by using the right hand side of the diagram.

The relation $P_\# f_\#=\bar{f}_\# p_\#$ implies $p_\# f_\#^n=\bar{f}_\#^n p_\#$. Since the rows in the diagram are exact, for each α in $\pi_1(E)$, there exists an element $\alpha^1\in\pi_1(E)$ such that $p_\#\alpha'=\alpha$.

Then we have
$$\begin{aligned} \bar{f}_\#^n(\alpha)\ \bar{f}_\#^n(\beta) &= \bar{f}_\#^n(p_\#\alpha')\bar{f}_\#^n(p_\#\beta')=p_\# f_\#^n(\alpha')p_\# f_\#^n(\beta') \\ &= p_\#(f_\#^n(\alpha')\cdot f_\#^n(\beta'))=p_\#(f_\#^n(\beta')\cdot f_\#^n(\alpha')) \\ &= p_\# f_\#^n(\beta')\ p_\# f_\#^n(\alpha')=\bar{f}_\#^n p_\#(\beta')\cdot \bar{f}_\#^n p_\#(\alpha') \\ &= \bar{f}_\#^n(\beta)\cdot \bar{f}_\#^n(\alpha). \end{aligned}$$

Thus $\bar{f}_\#^n(\pi_1(B))$ is an abelian subgroup of $\pi_1(B)$. We show now that f_b is eventually abelian by using the lift hand side of the diagram. The relation $f_\# i_\#=i_\# f_{b\#}$ implies the relation $f_\#^n i_\#=i_\# f_{b\#}^n$.

Then we have, for $\alpha,\beta\in\pi_1(p^{-1}(b))$, $i_\#(f_{b\#}^n(\alpha)f_{b\#}^n(\beta))=i_\#(f_{b\#}^n(\alpha\beta))=f_\#^n i_\#(\alpha\beta)$
$=f_\#^n(i_\#(\alpha)\cdot i_\#(\beta))=f_\#^n(i_\#\alpha))\cdot f_\#^n(i_\#(\beta))=f_\#^n(i_\#(\beta))\cdot f_\#^n(i_\#(\alpha))=f_\#^n(i_\#(\beta\alpha))$
$=i_\# f_{b\#}^n(\beta\alpha)=i_\#(f_{b\#}^n(\beta)\cdot f_{b\#}^n(\alpha))$

Since $i_\#$ is a monomorphism be have

$f_{b\#}^n(\alpha)\cdot f_{b\#}^n(\beta)=f_{b\#}^n(\beta)\cdot f_{b\#}^n(\alpha)$ *for* $\alpha,\beta\in\pi_1(p^{-1}(b))$. Using the result, we prove a product theorem for Nielsen numbers of a fiber-preserving map triple.

THEOREM 4.2. *Let $p: E \to B$ be an orientable Hurewicsz fibering with $E, B, p^{-1}(b)$, $b \in B$ connected, compact, metric ANR and $(f, \bar{f}, f_b)$ be a fiber-preserving map triple. If the inclusion map $i: p^{-1}(b) \subsetneq E$ incluces a monomorphism $i_{\#}: \pi_1(p^{-1}(b)) \to \pi_1(E)$ and if f is eventually abelian, then there is a constant $P(f)$ such that $N(f) \cdot P(f) = N(\bar{f}) \cdot N(f_b)$. (Compare* Theorem (4.10), IV *of*[9])

Proof. The inclusion map $i: p^{-1}(b) \to E$ induces the following commutative diagram

$$\begin{array}{ccc} H_1(p^{-1}(b)) & \xrightarrow{i_*} & H_1(E) \\ \downarrow f_{b*} & & \downarrow f_* \\ H_1(p^{-1}(b)) & \xrightarrow{i_*} & H_1(E). \end{array}$$

In turn, this diagram induces a homomorphism

$$i^*: H_1(p^{-1}(b))/(1-f_{b*})H_1(p^{-1}(b)) \longrightarrow H_1(E)/(1-f_{\#})(H_1(E))$$

Since $f_b{}^*$ is eventually abelian by Lemma (4.1), $R(f_b) = \mathrm{Ord}\ (H_1(p^{-1}(b))/(1-f_{b\#})H_1(p^{-1}(b)))$, [11], and $R(f) = H_1(E)/(1-f_{\#})(H_1(E))$.

Therefore i^* can be considered as a homomorphism between the Reidemeister classes of f_b and the Reidemeister classes of f.

Let $\Phi(f)$ and $\Phi(f_b)$ be the fixed point set of f and f_b, respectively, and $\Phi'(f)$ and $\Phi'(f_b)$ be the corresponding Nielsen fixed point classes. There are one-to-one into maps $\phi: \Phi'(f_b) \to R(f_b)$ and $\phi': \Phi'(f) \to R(f)$.

Thus we have

$$\Phi'(f_b) \xrightarrow[\phi]{1:1} H_1(p^{-1}(b))/(1-f_{b*})(H_1(p^{-1}(b))) \xrightarrow{i^*} H_1(E)/(1-f_*)(H_1(E)) \xrightarrow[\phi']{1:1} \Phi'(f).$$

Let $P(f)$ be the order of ker i^*. This number is independent of the choice of $F \in \Phi'(f)$ such that $p^{-1}(b) \cap F \neq \phi$ since i^* is a homomorphism. Thus for each $F \in \Phi'(f)$, $P(f)$ ts the cardinality of $\phi^{-1}\ i^{*-1}\phi'(F)$. That is, $P(f)$ is the number of Nielsen classes of f_b in $p^{-1}(b) \cap F$, Since F is essential if and only if $p(F)$ is essential in B, there lies precisely $N(f_b)/P(f)$ essential fixed point clusses of f for each essential class of $\bar{f}$: Therfne, we have $N(f) = N(\bar{f}) \cdot N(f_b)/P(f)$, since $\bar{f}$ hus $N(\bar{f})$ essential fixed point classes and $N(f_b)$ is independent of the cloice of $b \in B$.

We note that in Theorem (4.2) we may replace the condition that f is eventually abelian by the condition that there exist two positive integers m, n such that $f_{\#}{}^m(\pi_1(E)) \subset f_{\#}{}^n(J(f))$, where $J(f)$ is the Jiang subgroup of f.

5. On the equivariant maps

In this section, we prove a result that shows the conclusions of theorems (3.3) and (4.2) may be true even though not all the hypothesis are satisfied, in particular, when the fundamental group of the total space is non-abelian.

Let M be a compact, connected, orientable aspherical manifold with non-abelian fundamental group. If a compact, connected Lie group G acts effectively on M, then G has to be a total group T^k, [5]. Then the induced homomorphism $\alpha_{\#}: \pi_1(T^k) \to \pi_1(M)$ is a monomorphism and $\alpha_{\#}(\pi_1(T^k)$ lies in the center $Z(\pi_1(M))$ of $\pi_1(M)$. Let us denote the orbit space M/G by B and $p: M \to B$ be the projection map. It is well known that B is an aspherical manifold and $\mathfrak{J} = (M, p, B)$ become a singular fibering with the fiber T^k. Let $f: M \to M$ be a fiber-preserving map. Then f induces $\bar{f}$ and f_b as in the instruction such that $pf = \bar{f}p$. Assume $L(f) \neq 0$ and $f_\#$ is onto. Then the center $Z(\pi_1(M))$ is invariant under $f_\#$ and $\alpha_\#$ induces a homomorphism $\alpha^\#: \pi_1'(T^k) \to Z'(\pi_1(M)) \subset \pi_1'(M)$, and we have

$$\Phi'(f_b) \xrightarrow{1:1} \pi_1'(T^k) \xrightarrow{\alpha^\#} Z'(\pi_1(M)) \subset \pi_1'(M) \xleftarrow{1:1} \Phi'(f).$$

Where $\alpha^\#$ is induced by a map which asign, for each $F \in \Phi'(f)$, if $T^k \cap F \neq \phi$ the Nielsen classes of f_b in $T^k \cap F$ to F.

Thus, we have the following theorem using the argument used in the proofs of Theorems (3.3) and (4.2)

THEOREM 5.1. *Let (M, G) be a free action of a compact, connected Lie group G on a compact, connected, orientable aspherical manifold.*

Let $f: M \to M$ be a fiber-preserving map such that $L(f) \neq 0$. Then there exists a number P such that $N(f) \cdot P = N(\bar{f}) \cdot N(f_b), b \in B$.

Note that the fundamental group $\pi_1(M)$ need not be an abelian group.

If the induces homomorphian $f_\#: \pi_1(M) \to \pi_1(M)$ maps $\pi_1(M)$ into $Z(\pi_1(M))$, then the map between $\Phi'(f)$ and $\pi_1'(M)$ is one to one and onto. In this case $Z'(\pi_1(M))$ and $\pi_1'(M)$ have the same cardinality.

In a similar way, we can prove the following.

THEOREM 5.2. *Let (M, T^n) be a free action of the toral group T^n on a compact, connected, orientable aspherical manifald M. Let $f: M \to M$ be an eventually abelian fiber-preserving map with $L(f) \neq 0$, and the inclusion map $i: T^n \hookrightarrow M$ indue a monomorphism $i_\#: \pi_1(T^n) \to \pi_1(M)$. Then there exists a number $P(f)$ such that $N(f)P(f) = N(\bar{f})\ N(f_b)$, $b \in M/T^n$.*

In particular, if $f_\#(\pi_1(M))$ is an abelian subgroup of $\pi_1(M)$ and $f_\#\ (\pi_1(M)) \subset i_\#(\pi_1(T^n))$, then $i^\#$ is one to one, and hence $P(f) = 1$. Therefore we have the product relation $N(f) = N(\bar{f}) \cdot N(f_b)$.

We would like take this opportunity to say that Corollary (4.2) of [14] is incorrectly stated. Therefore the Corollary (4.2) and the last sentence of Example (4.3) should be omitted from the paper [14]. We thank Professor Bo-ju Jiang for pointing out this to us.

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