Bull. Korean Math. Soc.
Vol. 20, No. 1, 1983

A GENERALIZATION OF AN INEQUALITY OF LI AND ZHONG, AND ITS GEOMETRIC APPLICATION

Dedicated to the memory of Professor Dock Sang Rim.

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Let M be a n-dimensional compact Riemannian manifold with sectional curvature bounded below by one. Then Li and Zhong [3], and Li and Treibergs [4] proved that if the first eigenvalue of the Laplacian λ_1 is less than some universal constant and if $n\leq 4$, then M is diffeomorphic to the n-sphere S^n. The purpose of this paper is to prove this pinching theorem for all n with some extra condition. That is,

MAIN THEOREM. *Let M be a compact Reimannian manifold. Suppose the sectional curvature of M is greater than or equal to* 1. *There exists an universal constant $C>0$, depending on n, such that if $\lambda_1<n+c$ and when $\left|\frac{\inf u}{\sup u}\right|=k$, $k^2+k<1$ then M^n is diffeomorphic to S^n.*

The proof of this theorem depends on the following generalization of a gradient estimate obtained previously by Li and Zhong [3].

LEMMA. *Let M be a n-dmensional compact Riemannian manifold. Suppose* Ric $(M)\geq n-1$ *and if u is a non constant eigenfunction on M with* $\sup u=1$ *and* $\inf u=-k$ *where $k\leq 1$, then*

$$|\nabla u|^2\leq[\lambda-(n-1)](u+k)(2\alpha+k-u)$$

where $\alpha\geq 1$ and $\alpha\geq\frac{n-1}{\lambda-(n-1)}k$, and if $\lambda=n$ and $k=1$ then we take $\alpha=(n-1)k$.

Proof. We consider the function

$$G(x)=|\nabla u|^2(x)+[\lambda-(n-1)](u-\alpha)^2$$

defined on M. Let x_0 be a point in M such that G achieves its maximun value. At x_0, we have

$$G_i=0 \text{ for all } 1\leq i\leq n. \text{ Computation shows} \tag{1}$$

$$0=\sum_{j=1}^{n}u_ju_{ji}+[\lambda-(n-1)](uu_i-\alpha u_i) \text{ at } x_0. \tag{2}$$

If we assume $\nabla u\neq 0$ at x_0, and pick orthonormal frame at x_0 such that $e_1=\nabla u/|\nabla u|$, then (2) reduces to

$$0=u_{11}+[\lambda-(n-1)](u-\alpha) \text{ at } x_0. \tag{3}$$

*This research is partially supported by KOSEF.

By the maximality of G at x_0, we also have

$$0\geq\frac{\Delta G}{2}\geq u_{11}{}^2+\sum_{j,i}u_ju_{jii}+[\lambda-(n-1)](|\nabla u|^2-\lambda u^2+\lambda\alpha u) \tag{4}$$

By the Ricci calculus

$$\begin{aligned}\sum_{j,i}u_ju_{jii}&=\sum_{i,j}\mathrm{Ric}_{ij}u_iu_j+\sum_{i,j}u_ju_{iij}\\&\geq(n-1)|\nabla u|^2-\lambda|\nabla u|^2.\end{aligned} \tag{5}$$

Substituting into (4) yields

$$0\geq[\lambda-(n-1)]^2(u-\alpha)^2+[\lambda-(n-1)]\lambda u(\alpha-u) \tag{6}$$

However, by Lichnerowicz's theorem $\lambda\geq n$ and the assumption that $\sup u=1$, (6) reduces to

$$0\geq[\lambda-(n-1)](\alpha-u)+\lambda u=(n-1)u+\alpha[\lambda-(n-1)] \tag{7}$$

But this contradicts with the assumption that

$$\inf\ u=-k\geq-\left[\frac{\lambda-(n-1)}{n-1}\right]\alpha$$

unless

$$u(x_0)=-k=-\left[\frac{\lambda-(n-1)}{n-1}\right]\alpha$$

or

$$\nabla u(x_0)=0.$$

However both cases imply $\nabla u(x_0)=0$. hence

$$\begin{aligned}G(x)&\leq\max_{x\in M}\ [\lambda-(n-1)]\ (u-\alpha)^2\\&=[\lambda-(n-1)](u+k)(2\alpha+k-u)\end{aligned} \tag{8}$$

Proof of the MAIN THEOREM. As in Li and Zhong, we apply a theorem of Grove and Shiohama [1]. In Li and Yau [2], it was proved that when $\mathrm{Ric}(M)\geq1$, then the non constant eigenfunctions of M satisfy

$$\frac{|\nabla u|^2}{r^2-(w+u)^2}\leq\sup_{x\in M}\left[\frac{[\lambda-(n-1)](r^20(w+u)^2)+\lambda u(w+u)}{r^2}\right] \tag{9}$$

where w is any constant, and $r^2=\sup(w+u)^2$. If we set $w=\frac{k-1}{2}$, $\beta=\frac{1+k}{2}$ then (9) implies

$$\frac{|\nabla u|^2}{(u+k)(1-u)}\leq\frac{2\lambda}{1+k}. \tag{10}$$

Combining with our lemma we obtain

$$|\nabla u|^2\leq(u+k)\min\left\{\frac{2\lambda}{1+k}(1-u),\ [\lambda_1-(\mathrm{n}-1)]\,(2\alpha+k-u)\right\}$$

Let u_0 be the intersection of the two linear functions $\frac{2\lambda_1}{1+k}(1-u)$, and $[\lambda_1-(n-1)]$ $(2\alpha+k-u)$. One solves that

$$u_0=\frac{(n-1)(2+k)(1+k)-(k^2+(2+1)k+2-2)}{(1-k)_1+(n-1)(1+k)}. \tag{11}$$

If $\lambda_1=n$, then

$$u_0=\frac{2n-(1+k)(2\alpha+k)}{2n-(1+k)} \tag{12}$$

We claim that it is always possible to choose α such that M satisfies

$$\alpha\geq 1,\ \ \alpha\geq(n-1)k,\ \text{ and } 0<u_0<1 \tag{13}$$

if $k^2+k<\frac{2n}{2n-1}$.

First assume that $k<\frac{1}{n-1}$. Then it is trivial to see $\alpha=1$ satiesfies the first and third inequalities. If $k\geq\frac{1}{n-1}$, then leting $\alpha=(n-1)k$ we obtain the condition $2n-(1+k)(2(n-1)k+k)>0$, which is equivalent to $(1+k)k>\frac{2n}{2n-1}$. And it is satisfied especially when $(1+k)k>1$.

Therefore, by continuity, for $\lambda_1\leq n+C$ for small enough C, it is alway possible to find α satisfying the inequalities (13). Suppose r is a minimal geodesic which joins the maximum and the minimum point of u. Let r_1 and r_2 be the two segments of r which connects a point with value u_0 to the maximum and minimum points respectively. Then we obtain as in Li & Zhong

$$\frac{1+k}{2\lambda_1}\int_{u_0}^{1}\frac{du}{\sqrt{(u+k)(n-u)}}\leq l(r_1) \tag{14}$$

and

$$\frac{1}{\lambda_1-(n-1)}\int_{-k}^{u_0}\frac{du}{\sqrt{(u+k)(2\alpha+k-u)}}\leq l(r_2) \tag{15}$$

One computes easily that

$$\int_{u_0}^{1}\frac{du}{\sqrt{(u+k)(1-u)}}=\frac{\pi}{2}-\arcsin\frac{2u_0-1+k}{1+k}$$

and

$$\int_{-k}^{u_0}\frac{du}{\sqrt{(u+k)(2+k-u)}}=\frac{\pi}{2}+\arcsin\frac{u_0-\alpha\alpha+k}{1+k}$$

Especially when $\lambda_1=n$, and $u_0=\frac{2n-(1+k)(2\alpha+k)}{2n-(1+k)}$

we have

$$d\geq\frac{1+k}{2n}\left(\frac{\pi}{2}-\text{arc sin}\frac{2n-4\alpha-(3k-1)}{2n-(1+k)}\right)+\frac{\pi}{2}-\text{arc sin}\frac{k^2+(\alpha+1)k+2\alpha n+\alpha-2n}{(2n-(1+k))(\alpha+k)} \tag{16}$$

The right hand side of (16) is bigger $\frac{\pi}{2}$ if $k\leqslant\frac{1}{n-1}$ as in Li and Zhong if we choose $\alpha=1$. If $k\geqq\frac{1}{n-1}$, let $\alpha=(n-1)k$. Then the right hand side of (16) becomes

$$\frac{1+k}{2n}\left(\frac{\pi}{2}-\text{arc sin}\frac{2n-4nk+k+1}{2n-(1+k)}\right)$$
$$+\frac{\pi}{2}-\text{arc sin}\frac{k^2+(2n-1)k-2}{(2n-(1+k))k}.$$

One readily checks that it takes minimum value at $k=\frac{1}{n-1}$, in which case

$$d\geq\frac{1}{\sqrt{2(n-1)}}\left(\frac{\pi}{2}-\text{arc sin}\frac{2n^2-5n}{2n^2-3n-2}\right)+\frac{\pi}{2}-\text{arc sin}\frac{1}{2n-3}>\pi/2.$$

This concludes the proof of the theorem.

References

1. Grove, K. and K. Shiohama, *A generalized sphere theorem*, Ann. of Math. **106**, 201-211 (1977).
2. Li. P and S. T. Yau, *Estimates of eigenvalues of a compact Riemannian Manifold*, Proc. Sym. Pure Math. **36**, 205-239 (1980)
3. Li, P and J. Q. Zhong, *Pinching Theorem for the first Eigenvalue on Positively Curved Manifolds*, Invent. Math **65**, 221-225 (1981)
4. Li, P and A. Treibergs, *Pinching Theorem for the first eigenvalue on Positively curved Four-manifolds*, Invent. Math. **66**, 35-38 (1982)
5. Lichnerowicz, A, *Geometrie des groupes de transformations*, Paris, Dunod (1958)

REMARK: Recently C. B. Croke (Invent. math. 68, 253-256(1982)) solved the eigenvalue pinching problem up to homeomorphism. But it remins unsolved to show that it is actually diffeomorphic to the standard sphere.

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