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THE FINITE DIMENSIONAL PRIME RINGS

Dedicated to the memory of my friend and teacher Dock S. Rim

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1. Introduction

If R is a ring and M is a right (or left) R-module, then M is called a faithful R-module if, for some a in R, $x \cdot a=0$ for all $x \in M$ then $a=0$. In [4], R.E. Johnson defines that M is a prime module if every non-zero submodule of M is faithful. Let us define that M is of prime type provided that M is faithful if and only if every non-zero submodule is faithful. We call a right (left) ideal I of R is of prime type if R/I is of prime type as a R-module. This is equivalent to the condition that if $xRy \subseteq I$ then either $x \in I$ or $y \in I$ (see [5 : 3.1]). It is easy to see that in case R is a commutative ring then a right or left ideal of a prime type is just a prime ideal. We have defined in [5], that a *chain* of right ideals of prime type in a ring R is a finite strictly increasing sequence $I_0 \subset I_1 \subset \ldots \subset I_n$; the length of the chain is n. By the *right dimension* of a ring R, which is denoted by $\dim_r R$, we mean the supremum of the length of all chains of right ideals of prime type in R. It is an integer $\geqq 0$ or ∞. The left dimension of R, which is denoted by $\dim_l R$ is similarly defined. It was shown in [5], that $\dim_r R=0$ if and only if $\dim_l R=0$ if and only if R modulo the prime radical is a strongly regular ring. By "a strongly regular ring", we mean that for every a in R there is x in R such that $axa=a=a^2x$. It was also shown that R is a simple ring if and only if every right ideal is of prime type if and only if every left ideal is of prime type. In case, R is a (right or left) primitive ring then $\dim_r R$ $=n$ if and only if $\dim_l R=n$ if and only if $R \cong D_{n+1}$, $n+1$ by $n+1$ matrix ring on a division ring D. In this paper, we establish the following results:

(a) If R is a prime ring and $\dim_r R=n$ then either R is a right Ore domain such that every non-zero right ideal of a prime type contains a non-zero minimal prime ideal or the classical ring of right quotients is isomorphic to $m \times m$ matrix ring over a division ring where $m \leq n+1$.

(b) If R is a prime ring and $\dim_r R=n$ then $\dim_l R=n$ if $\dim_l R<\infty$.

(c) Let R be a principal right and left ideal domain. If $\dim_r R=1$ then R is an unique factorization domain.

2. To give the reader a feel for the theory, we start with the various examples of finite dimensional prime rings.

2.1 EXAMPLE. Let Z be the ring of integers. Let R be the 2×2 matrix ring over Z. Since R is a prime ring, by [5 : 2.10] every annihilator right ideal is of prime type.

If $x \in R$, let $x^r=\{y \in R \mid xy=0\}$.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^r \neq 0$ then $ad=bc=0$. Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^r$ is one of the following: $\begin{pmatrix} 0 & 0 \\ Z & Z \end{pmatrix}$, $\begin{pmatrix} Z & Z \\ 0 & 0 \end{pmatrix}$, $\left\{ \begin{matrix} b\,x & b\,y \\ -a\,x & -a\,y \end{matrix} \middle| x, y \in Z \right\}$ and $\left\{ \begin{matrix} d\,x & d\,y \\ -c\,x & -c\,y \end{matrix} \middle| x, y \in Z \right\}$. It is also easy to check that if p and q are prime numbers, $\begin{pmatrix} pZ & pZ \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ qZ & qZ \end{pmatrix}$, $\begin{pmatrix} pZ & pZ \\ qZ & qZ \end{pmatrix}$, $\begin{pmatrix} Z & Z \\ qZ & qZ \end{pmatrix}$, $\begin{pmatrix} pZ & pZ \\ Z & Z \end{pmatrix}$ are ideals of prime type. In fact, any right ideal of prime type must be one of these types. The longest chain of right ideals of prime type is $0 \subset \begin{pmatrix} pZ & pZ \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} Z & Z \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} Z & Z \\ qZ & qZ \end{pmatrix}$ Therefore $\dim_r R=3=\dim_l R$.

2.2 Example. Let D be the ring of real quarternions. Let $R=D[X]$, the polynomials over D where $dX=Xd$ for all $d \in D$ Then right and left division algorithm holds. Hence R is a right and left principal ideal ring. Now it is easy to see that if fR, for some $f \in R$, is a two sided ideal then f is in the center of R. Hence if fR is a prime ideal then $f(X)$ is an irreducible real polynomial. It is not difficult to show that if f is an irreducible element in R then fR is a maximal right idieal and fR contains a non-zero two sided ideal. Hence R is not a primitive ring. Therefore $fR \supset P \supset (0)$ where P is the primitive ideal which is contained in fR, is the longest possible chain of right ideals of a prime type. Hence $\dim_r R=2=\dim_l R$. It is interesting to note that for an arbitrary division ring D, $\dim_r D[X]=1$ if and only if D is a field.

2.3 Example. Let C be the field of complex numbers. There is a monomorphism θ of C into C which is not an epimorphism. Let $R=C[X, \theta]$ denote the set of all formal polynomials in the indeterminate X with coefficients in C written on the right of power of X. Define equality and addition in $C[X, \theta]$ as usual but define a multiplication by assuming the distributive laws and the rule

$$a \cdot X = X \cdot \theta(a) \text{ for each } a \in C.$$

The division algorithm holds on the right side. So R is a principal right ideal domain. However, if $\alpha \in C \backslash \theta(C)$ then $R(X-\alpha) \cap RX^2=(0)$. So it is not a left Ore domain. Consider a maximal right ideal $(X-1)R$. Let $g \in (X-1)R$ such that $fg \in (X-1)R$ for all $f \in R$. If $g \neq 0$ then $g=(X-1)\ (X^h+a_1X^{h-1}+\ldots+a_n)$ for some non-negative integer n and a_i in C. If b is a complex number $bg=(X\theta(b)-b)\ (X^h+a_1X^{h-1}+\ldots+a_n)$. Hence unless $\theta(b)=b$, $bg \notin (X-1)R$. This means that g must be 0. Therefore, $R/(X-1)R$ is a faithful irreducible R-module and R is a primitive ring. If $\dim_r R<\infty$ then either R is a division ring or $n \times n$ matrix ring over a division ring with $n \geq 2$, in view of [5 : 4.3] this is not possible. Thus $\dim_r R=\infty$. For more interesting examples of this type we refer the readers to [3].

3. Recall that a ring R is called a prime ring if zero ideal is of a prime type

For each right (left) ideal I, let $s(I)$ denote the largest two sided ideal of R which is contained in I. In [5], it was shown that $s(I)$ is a prime ideal if I is a right ideal of a prime type. Furthermore, if $\{I_\alpha\}_{\alpha \in A}$ is a family of right ideals prime type in a prime ring such that each $s(I_\alpha)=0$ than $\bigcap_{\alpha \in A} I_\alpha$ is also a right ideal of a prime type. A non-zero right (left) ideal of R is called *essential* if it has a non-zero intersection with each non-zero right (left) ideal (Refer [1]). A right ideal I of R is *relatively complemented* if $I \neq 0$ and if there exists a non-zero right ideal J of R with $I \cap J = 0$. The set of right ideals which contains J and have zero intersection with I, has maximal elements by Zorn's lemma. These maximal elements are the *relative complements* of I. A right ideal of R which is a relative complement of a non-zero right ideal is called a complement right ideal (See [1]). In the sequel, unless otherwise stated, all rings are prime.

3.1. LEMMA. *Assume* $\dim_r R < \infty$. *If I is an essential right ideal of a prime type in R then* $s(I) \neq 0$. *Conversely, if* $s(I) \neq 0$ *then I is essential.*

Proof. Let $M = R/I$. Suppose that $s(I) = 0$. Then M is a faithful right R-module. For each $m \in M$, let $m^{\perp} = \{x \in R \mid mx = 0\}$. If $m \neq 0$, then $s(m^{\perp}) = 0$. By [5 : 2.10], $m^{\perp}$ is a right ideal of a prime type. Hence for any non-zero family $\{m_\alpha\}_{\alpha \in \Lambda}$ in M, $\bigcap_{\alpha \in \Lambda} m_\alpha^{\perp}$ is a right ideal of a prime type by [5 : 3.2]. Since $\dim_r R < \infty$, there is a positive integer n and a finite number of elements $m_1, m_2 \ldots, m_n$ in M such that $\bigcap_{i=1}^{n} m_i^{\perp} \cap m^{\perp} = \bigcap_{i=1}^{n} m_i^{\perp}$ for all $m \in M$. Since each $m_i^{\perp}$ is essential, $\bigcap_{i=1}^{n} m_i^{\perp} \neq 0$ and $s(I)$ contains $\bigcap_{i=1}^{n} m_i^{\perp} \neq 0$. Thus $s(I) \neq 0$ and M is not faithful. This is a contradiction. In a prime ring, non-zero ideal is essential. Hence if $s(I) \neq 0$ then I is essential.

3.2. LEMMA. *If* $\dim_r R < \infty$, *R is a right Goldie ring.*

Proof. Recall that R is a right Goldie ring if it satisfies the maximum condition on annihilator right ideals and complement right ideals (Refer [1]).

Since annihilator right ideals and complement right ideals are of prime type by [5 : 2.10] and by [5 : 3.5], a finite dimensionality of R implies that R is a right Goldie ring.

3.3. LEMMA. *Suppose that* $\dim_r R < \infty$. *If I is a right ideal of a prime type which is not essential then I is a complement right ideal.*

Proof. Since I is not essential, there is a non-zero right ideal J which is a relative complement of I. Let $\bar{I}$ be a relative complement of J which contains I. If $I \subsetneq \bar{I}$, then for each $x \in \bar{I} \setminus I$, $(I : x) = \{r \in R \mid xr \in I\}$ is an essential and it is a right ideal of a prime type by ([: 2.10]). Since $\dim_r R < \infty$, there exist a finite number of elements $x_1, x_2, \ldots, x_n$ in $\bar{I} \setminus I$ such that $\left[\bigcap_{i=1}^{n} (I : x_i)\right] \cap (I : x) = \left[\bigcap_{i=1}^{n} (I : x_i)\right]$ for all $x \in \bar{I}$. Since for each x_i, $s((I : x)) \neq 0$ by 3.1 and $s\left(\bigcap_{i=1}^{n} (I : x_i)\right) = J \neq 0$, $\bar{I}J \subseteq I$. Now I is a right ideal of a prime type and $\bar{I} \not\subseteq I$. Therefore, $J \subseteq I$. This means that $s(I) \neq 0$ and I is essential. This is impossible. Thus $I = \bar{I}$ and I is a complement right ideal.

3.4. LEMMA. *Assume that* $\dim_r R < \infty$. *Let* $0 \subset I_1 \subset I_2 \subset \ldots \subset I_k$ *be the longest chain of right ideals of a prime type, which are not essential. Then if J is an essential right ideal which contains a maximal complement right ideal then the length of the longest chain of the non-essential right ideals of prime type in J is* $k+1$.

Proof. We note that R is a right Goldie prime ring. Observe that I_1 is a minimal complement and I_k is a maximal complement. Since the Goldie dimension of R is equal to the Goldie dimension of J as R-module, the assertion is true (Refer [1 2]).

3.5. THEOREM. *Let* $\dim_r R=n$ *for some positive integer* n.
Then $\dim_l R=n$ *if* $\dim_l R<\infty$.

Proof. By 3.2, R is a right and left Goldie ring since $\dim_r R<\infty$ and $\dim_l R<\infty$. Therefore, the right Goldie dimension of R is equal to the left Goldie dimension of R. Hence if J is an essential right ideal of a prime type and L is an essential left ideal of a prime type then by 3.4, the length of the longest chain of essential right ideals of prime type which is contained in J is precisely equal to the length of the longest chain of essential left ideals of a prime type in L. Let $0\subset I_1\subset I_2\subset\ldots\subset I_n$ be the longest chain of right ideals of a prime type in R. If $n=0$ then $\dim_r R=\dim_l R$ by [5 : 2.9]. Let I_k be the first essential right ideal of a prime type in the chain. Then $I_k=s(I_k)$ by 3.1 and $R/s(I_k)$ is a prime ring such that $\dim_r R/s(I_k)=n-1$ and $\dim_l R/s(I_k)<\infty$. Hence by the inductive assumption, $\dim_l R/s(I_k)=n-1$. Thus $\dim_r R=n=\dim_l R$.

3.6. THEOREM. *If* $\dim_r R=n$ *for some positive integer* n, *then either* R *is a right Ore domain such that every non-zero right ideal of a prime type contains a non-zero minimal prime ideal or the classical ring of right quotions in isomorphic to* $m\times m$ *matrix ring over a division ring where* $m\leq n+1$.

Proof. R is a right Goldie ring. Let U be a uniform right ideal of R. Then there exist $u_1, u_2, \ldots, u_m$ in U such that

$$\bigcap_{i=1}^{m}(0 : u_i)=0 \text{ but } \bigcap_{\substack{i=1\\ i\neq j}}^{m}(0 : u_i)\neq 0 \text{ for each } j. \text{ If } m=1,$$

then R is a right Ore domain in which every non-zero right ideal is essential. Hence by 3.3, every non-zero right ideal of a prime type contains a non-zero prime ideal, hence it contains a non-zero minimal prime ideal. Since $\bigcap_j (0 : u_j)$ is a right ideal of a prime type, $m\leq n+1$. The rest of the theorem follows from Goldie's theorems in [2 : 37] and [2 : 4].

4. If R is a right or left noetherian domain then every non-unit has an irreducible factor.

For if a in R does not have an irreducible factor, we may form a strictly ascending chain of right ideals:

$$aR\subset a_1R\subset a_2R\subset\ldots, \text{ where}$$
$$a=a_1b_1=(a_2b_2)b_1=((a_3b_3)b_2)b_1=\ldots \text{ for some } a_i, b_i \text{ in } R.$$

R is called a principal right (left) ideal ring if every right (left) ideal is generated by one element. (See [3]).

4.1. THEOREM. *Let* R *be a principal right and left ideal domain. If* $\dim_r R=1$ *then* R *is an unique factorization domain and for any* x, y *in* R, $x\cdot y=\varepsilon y\cdot x$ *for some unit*

ε.

Proof Since R is a principal right and left ideal domain, every maximal right or left ideal is generated by an irreducible element. Let I be a maximal right ideal of R. Since I is essential, $s(I)\neq 0$ by 3.1. Since $\dim_r R=1$, $s(I)=I$.

Let $I=pR$ for some irreducible element p in R. Then Rp is a maximal left ideal which is contained in I. Hence $pR=Rp$. Therefore for any x and y, $px=x'p$ for some x' in R and $yp=py'$ for some y' in R.

Now we claim that every non-unit a in R is a product of a finite number of irreducible elements. Certainly, $a=p_1b_1$ for some irreducible p_1 and an element b_1 in R. $b_1=p_2b_2$ for some irreducible p_2 and an element b_2 in R unless b_1 is an unit. Now $a=p_1p_2b_2=p_2'p_1b_2=p_2'b_2'p_1=\dots$ and we have a strictly ascending chain of right ideals $aR\subset p_2'b_2'R\subset\dots$, unless b_2 is a finite product of irreducible elements. Now let p be an irreducible element and a, b in R such that $a\cdot b=pc$ for some c in R. If p does not factor a then there exist x and y in R, such that $xa+yp=1$ and $xab+ypb=b=xab+yb'p=b$ for some b' in R. Hence p divides b.

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